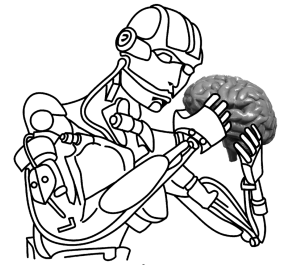
CS545—Contents VI

- Control Theory II
  - Linear Stability Analysis
  - Linearization of Nonlinear Systems
  - Discretization
- Reading Assignment for Next Class
  - See http://www-clmc.usc.edu/~cs545

Stability Analysis



- Given the control system

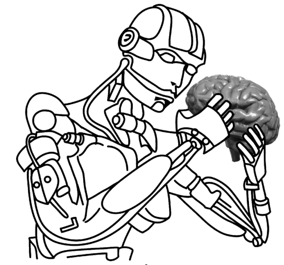
$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}) \quad \text{or} \quad \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

- How can we the show that a particular choice of a controller generates a stable control system?

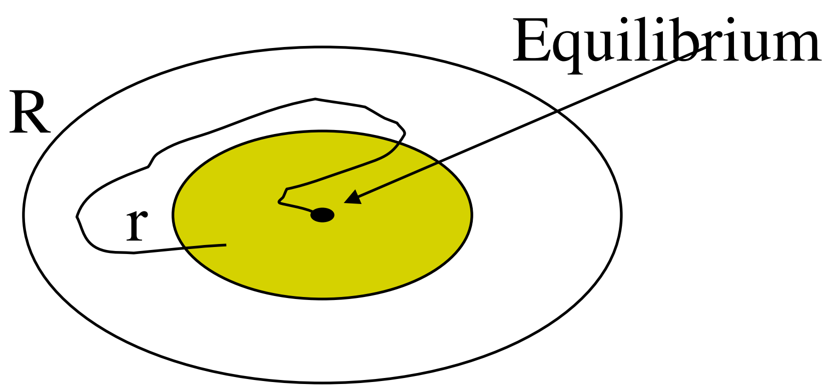
- In order to get started, consider whether the generic dynamical system is stable:

$$\dot{\mathbf{x}} = f(\mathbf{x}) \quad \text{or} \quad \dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

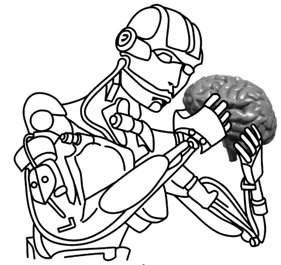
Equilibrium Points and Stability



- Definition of an Equilibrium Point
  - A state x is an equilibrium state (or equilibrium point) of the system if once $x(t)$ is equal to x, it remains equal to x for all future time.
  - Mathematically, this means:
- Definition of Stability
  - An equilibrium state x is said to be stable, if, for any $R>0$, there exists $r>0$, such that if $||x(0)||<r$,then $||x(t)||<R$ for all $t \geq 0$. Otherwise, the equilibrium point is unstable.



Linear Stability Analysis (Local Stability Analysis)



- What is needed at the outset:
  - The system model (linear or nonlinear)
  - An equilibrium point
  - The linearization of the system about the equilibrium point
- Then, we have the linear(ized) system:

$$\dot{\tilde{\mathbf{x}}} = \mathbf{A}\big|_{\mathbf{x}=\mathbf{x}^*}\,\tilde{\mathbf{x}} \qquad \text{where} \quad \tilde{\mathbf{x}} = \left(\mathbf{x} - \mathbf{x}^*\right)$$

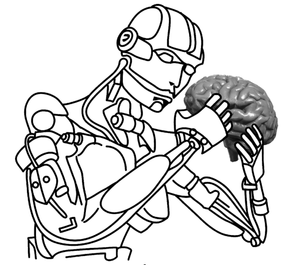
- This system is stable if and only if:

$$REAL\left(eig(A)\right) < 0$$

- The complete stability definitions are:

	continuous system	discrete system
stable	$REAL\left(eig(A)\right) < 0$	$\lVert eig(A) \rVert < 1$
marginally stable	$REAL\left(eig(A)\right) = 0$	$\lVert eig(A) \rVert = 1$
unstable	$REAL\left(eig(A)\right) > 0$	$\lVert eig(A) \rVert > 1$

Example: Stability of Pendulum



- The nonlinear equations of a controlled pendulum were derived in Lecture III to be:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -\frac{g}{l}\sin(x_2) \\ x_1 \end{pmatrix} + \tau \begin{pmatrix} 1/ml^2 \\ 0 \end{pmatrix}$$

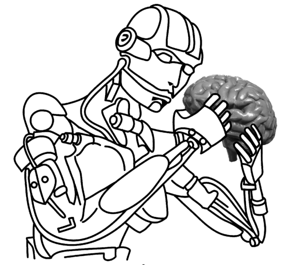
- Just consider the system without control input:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -\frac{g}{l}\sin(x_2) \\ x_1 \end{pmatrix}$$

- What are the equilibrium points?

$$0 = \begin{pmatrix} -\frac{g}{l}\sin(x_2) \\ x_1 \end{pmatrix} \quad \Rightarrow \quad x_1 = 0,\ x_2 = 0 \text{ or } \pi$$

Example: Stability of Pendulum (cont'd)



- Are the equilibrium points locally stable?
- Linearization:
  - Taylor Series Expansion about the equilibrium point:

    Given: $\dot{\mathbf{x}} = f(\mathbf{x})$

    then a First Order Taylor Series Expansion about

    a point x_0 is:

    $$f(\mathbf{x}) \approx f(\mathbf{x}_0) + \left.\frac{\partial f}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}_0} (\mathbf{x} - \mathbf{x}_0)$$

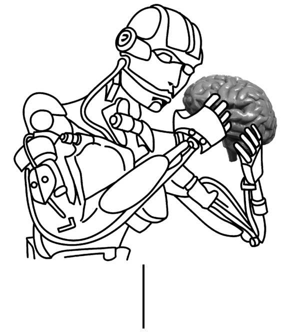
  - For the pendulum example:

$$f(\mathbf{x}) = \begin{pmatrix} -\frac{g}{l}\sin(x_2) \\ x_1 \end{pmatrix}$$

$$\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} = \begin{pmatrix} 0 & -\frac{g}{l}\cos(x_2) \\ 1 & 0 \end{pmatrix};$$

$$\left.\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}=0} = \begin{pmatrix} 0 & -\frac{g}{l} \\ 1 & 0 \end{pmatrix}; \quad \left.\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}=\pi} = \begin{pmatrix} 0 & \frac{g}{l} \\ 1 & 0 \end{pmatrix}$$

Example: Stability of Pendulum (cont'd)



- Eigenvalues:
  - Case 1:

$$eig\begin{pmatrix} 0 & -\frac{g}{l} \\ 1 & 0 \end{pmatrix}: \quad \lambda^2 + \frac{g}{l} = 0 \quad \Rightarrow \lambda_{1,2} = \pm j\sqrt{\frac{g}{l}}$$

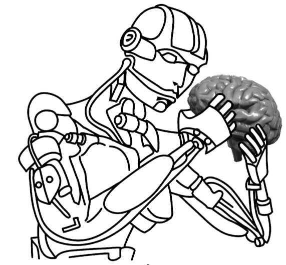
$$\Rightarrow \text{ marginally stable}$$

  - Case 2:

$$eig\begin{pmatrix} 0 & \frac{g}{l} \\ 1 & 0 \end{pmatrix}: \quad \lambda^2 - \frac{g}{l} = 0 \quad \Rightarrow \lambda_{1,2} = \pm\sqrt{\frac{g}{l}}$$

$$\Rightarrow \text{ unstable}$$

Make the Pendulum Stable: Add Dissipation



- In order to be stable, a system MUST be able to lose energy somehow
  - Add viscous friction to pendulum

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix}=\begin{pmatrix}-\frac{g}{l}\sin(x_2)-bx_1\\ x_1\end{pmatrix}$$

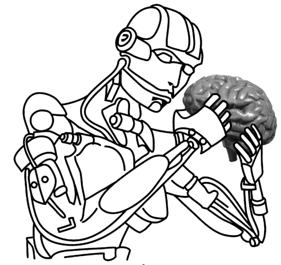
  - Linearized System

$$\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}=\begin{pmatrix}-b & -\frac{g}{l}\cos(x_2)\\ 1 & 0\end{pmatrix};$$

$$\left.\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}=0}=\begin{pmatrix}-b & -\frac{g}{l}\\ 1 & 0\end{pmatrix};\quad \left.\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}=\pi}=\begin{pmatrix}-b & \frac{g}{l}\\ 1 & 0\end{pmatrix}$$

$$\Rightarrow \lambda^2+b\lambda\pm\frac{g}{l}=0 \quad \Rightarrow \quad \lambda_{1,2}=\frac{1}{2}\left(-b\pm\sqrt{b^2\mp 4\frac{g}{l}}\right)$$

The Controlled Pendulum



$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -\frac{g}{l}\sin(x_2) \\ x_1 \end{pmatrix} + \tau \begin{pmatrix} 1/ml^2 \\ 0 \end{pmatrix}$$

- Assume a PD controller:

$$\tau = k_p\left(x_{2,d} - x_2\right) + k_D\left(x_{1,d} - x_1\right)$$

$$= \begin{pmatrix} k_D & k_P \end{pmatrix} \left(\begin{pmatrix} x_{1,d} \\ x_{2,d} \end{pmatrix} - \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = \mathbf{K}\left(\mathbf{x}_d - \mathbf{x}\right)$$

- … and apply to the system

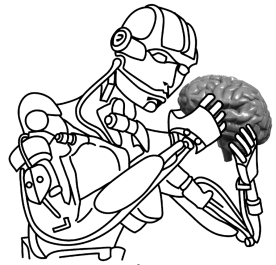
$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -\frac{g}{l}\sin(x_2) \\ x_1 \end{pmatrix} + \begin{pmatrix} 1/ml^2 \\ 0 \end{pmatrix} \mathbf{K}\left(\mathbf{x}_d - \mathbf{x}\right)$$

- Equilibrium Points:

$$0 = \begin{pmatrix} -\frac{g}{l}\sin(x_2) \\ x_1 \end{pmatrix} + \begin{pmatrix} 1/ml^2 \\ 0 \end{pmatrix} \mathbf{K}\left(\mathbf{x}_d - \mathbf{x}\right)$$

$$\Rightarrow x_1 = 0, \quad -\frac{g}{l}\sin(x_2) + 1/ml^2 \, k_P\left(x_{2,d} - x_2\right) = 0$$

The Controlled Pendulum (cont' d)



- Linearize the system

$$\begin{pmatrix} 0 & -\frac{g}{l}\cos(x_{2,0}) \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1/ml^2 \\ 0 \end{pmatrix} \mathbf{K} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

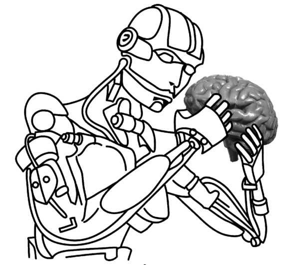
$$= \begin{pmatrix} -k_D/ml^2 & -\frac{g}{l}\cos(x_{2,0}) - k_P/ml^2 \\ 1 & 0 \end{pmatrix}$$

- Eigenvalues can be determined as before
  - Note: D-controller makes the system stable!!!!

$$\begin{pmatrix} -k_D/ml^2 & -\frac{g}{l}\cos(x_{2,0}) - k_P/ml^2 \\ 1 & 0 \end{pmatrix}, \text{ assume } m = l = 1$$

$$\lambda_{1,2} = \frac{1}{2}\left(-k_D \pm \sqrt{k_D^2 - 4\left(k_P + g\cos(x_{2,0})\right)}\right)$$

Discretization



- In real implementations, time is discrete, and continuous systems need to be discretized in order to examine stability

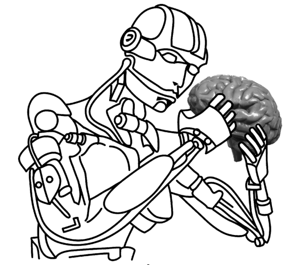
$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\frac{\mathbf{x}^{n+1} - \mathbf{x}^{n}}{\Delta t} = \mathbf{A}\mathbf{x}^{n} + \mathbf{B}\mathbf{u}^{n}$$

$$\mathbf{x}^{n+1} = \Delta t\mathbf{A}\mathbf{x}^{n} + \Delta t\mathbf{B}\mathbf{u}^{n} + \mathbf{x}^{n}$$

$$\mathbf{x}^{n+1} = (\Delta t\mathbf{A} + I)\mathbf{x}^{n} + \Delta t\mathbf{B}\mathbf{u}^{n}$$

Discretized Pendulum Stability



- The discretized linearized system becomes

$$\begin{pmatrix} 1 & -\frac{g}{l}\cos\left(x_{2,0}\right)\Delta t \\ \Delta t & 1 \end{pmatrix} + \begin{pmatrix} \Delta t / ml^2 \\ 0 \end{pmatrix} \mathbf{K} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \Delta t k_D / ml^2 & -\frac{g\Delta t}{l}\cos\left(x_{2,0}\right) - \Delta t k_P / ml^2 \\ \Delta t & 1 \end{pmatrix}$$

- Example: equilibrium state is zero and desired state is zero

$$\begin{pmatrix} 1 - \Delta t k_D / ml^2 & -\frac{g\Delta t}{l} - \Delta t k_P / ml^2 \\ \Delta t & 1 \end{pmatrix}$$

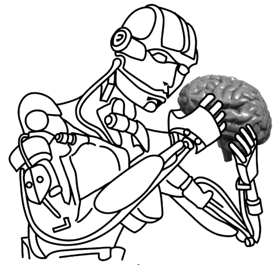
for simplicity, assume $m = l = 1$

$$\begin{pmatrix} 1 - \Delta t k_D & -\Delta t\left(g + k_P\right) \\ \Delta t & 1 \end{pmatrix}$$

$$\lambda^2 + \lambda\left(-2 + \Delta t k_D\right) + 1 + \Delta t^2\left(k_P + g\right) + \Delta t k_P = 0$$

$$\lambda_{1,2} = \frac{1}{2}\left(2 - \Delta t k_D \pm \sqrt{\left(2 - \Delta t k_D\right)^2 - 4\left(1 + \Delta t^2\left(k_P + g\right) + \Delta t k_P\right)}\right)$$

Eigenvalues as a function of gains



Note: inappropriate gains can make a naturally stable equilibrium point unstable