CS 545 Lecture 6
Coordinate Transformations

- Position
- Rotation
- Composition of Rotations
- Rotations around Arbitrary Axis
- Euler Angles
- Homogeneous Transformations

http://robotics.usc.edu/~aatrash/cs545

Introduction

- Robots composed of *links* and *joints*
  - Possibly with manipulator at end
- Need to determine position of tip as function of joints (Kinematics)
- Select joint angles to achieve configuration (Inverse Kinematics)
- Series of pendulums

Position

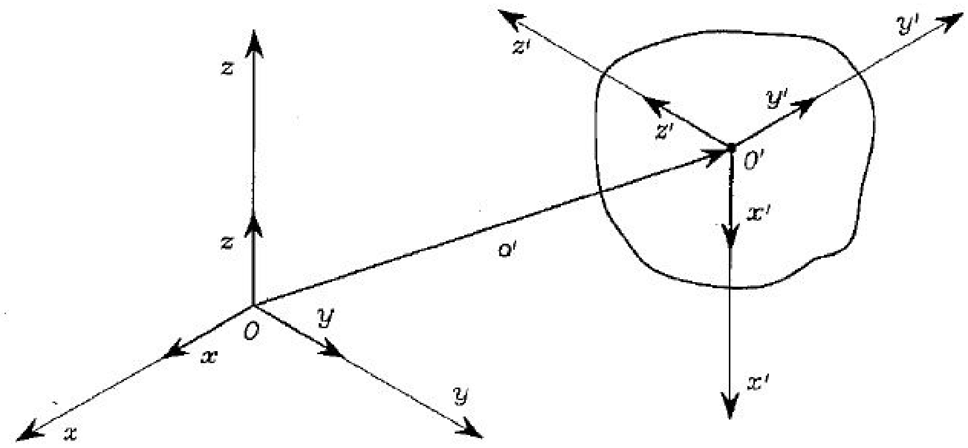


FIGURE 2.1
Position and orientation of a rigid body.

- Rigid body defined by *position* and *orientation* relative to *reference frame (**x,y,z**)* where ***x,y,z*** are unit vectors.
- *Position*:

$$\boldsymbol{o}' = o'_x \boldsymbol{x} + o'_y \boldsymbol{y} + o'_z \boldsymbol{z}, \qquad o' = \begin{bmatrix} o'_x \\ o'_y \\ o'_z \end{bmatrix}$$

- *Orientation*: Attach frame to object (*x’,y’,z’) (see Figure above)*

Rotation

- *Rotation* defined between the two frames

$$
\begin{aligned}
\boldsymbol{x}' &= x'_x \boldsymbol{x} + x'_y \boldsymbol{y} + x'_z \boldsymbol{z} \\
\boldsymbol{y}' &= y'_x \boldsymbol{x} + y'_y \boldsymbol{y} + y'_z \boldsymbol{z} \\
\boldsymbol{z}' &= z'_x \boldsymbol{x} + z'_y \boldsymbol{y} + z'_z \boldsymbol{z}.
\end{aligned}
$$

- *Rotation Matrix*

$$
\boldsymbol{R} = \begin{bmatrix} \boldsymbol{x}' & \boldsymbol{y}' & \boldsymbol{z}' \end{bmatrix} = \begin{bmatrix} x'_x & y'_x & z'_x \\ x'_y & y'_y & z'_y \\ x'_z & y'_z & z'_z \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}'^T\boldsymbol{x} & \boldsymbol{y}'^T\boldsymbol{x} & \boldsymbol{z}'^T\boldsymbol{x} \\ \boldsymbol{x}'^T\boldsymbol{y} & \boldsymbol{y}'^T\boldsymbol{y} & \boldsymbol{z}'^T\boldsymbol{y} \\ \boldsymbol{x}'^T\boldsymbol{z} & \boldsymbol{y}'^T\boldsymbol{z} & \boldsymbol{z}'^T\boldsymbol{z} \end{bmatrix}
$$

Elementary Rotations

- Assume reference and rotated frame share origin (relaxed later)
- Rotations around one of the coordinate axis.
- Rotating around z-axis:

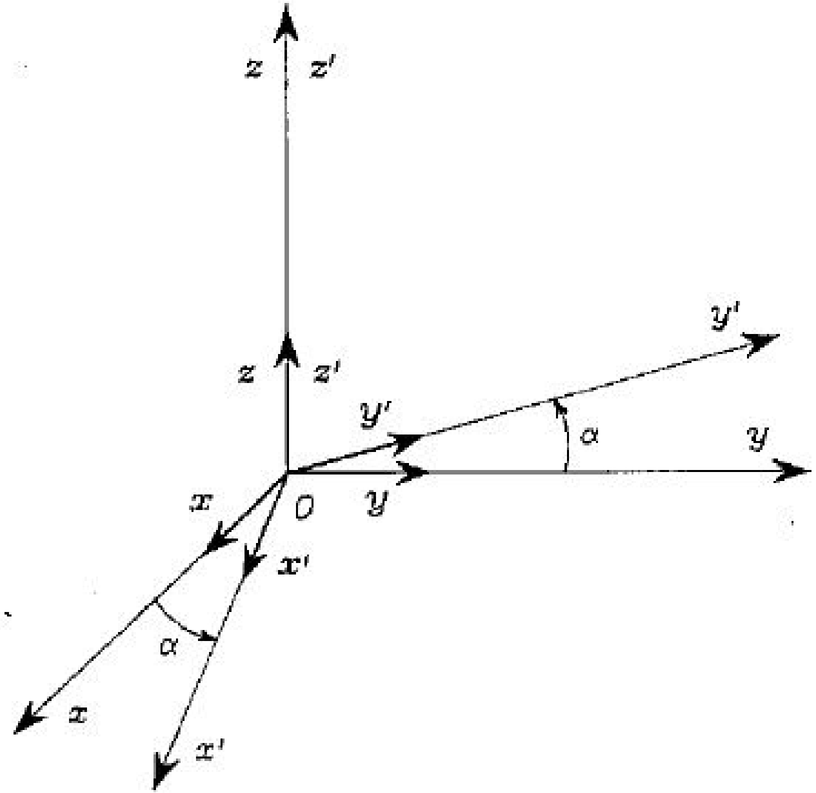
$$x' = \begin{bmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{bmatrix} \qquad y' = \begin{bmatrix} -\sin\alpha \\ \cos\alpha \\ 0 \end{bmatrix} \qquad z' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Elementary Rotations

$$\boldsymbol{R}_z(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{R}_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

$$\boldsymbol{R}_x(\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}$$



$$\boldsymbol{R}_k(-\vartheta) = \boldsymbol{R}_k^T(\vartheta)$$

Representation of Vectors

- Point in *o′*: $p' = \begin{bmatrix} p'_x \\ p'_y \\ p'_z \end{bmatrix}$

- What does that point look like in *o*?
- Let *R* be the rotation between *o* and *o′*.
- Transformation: $p = Rp'$ $p = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$

- Transform back: $p' = R^T p$

Example

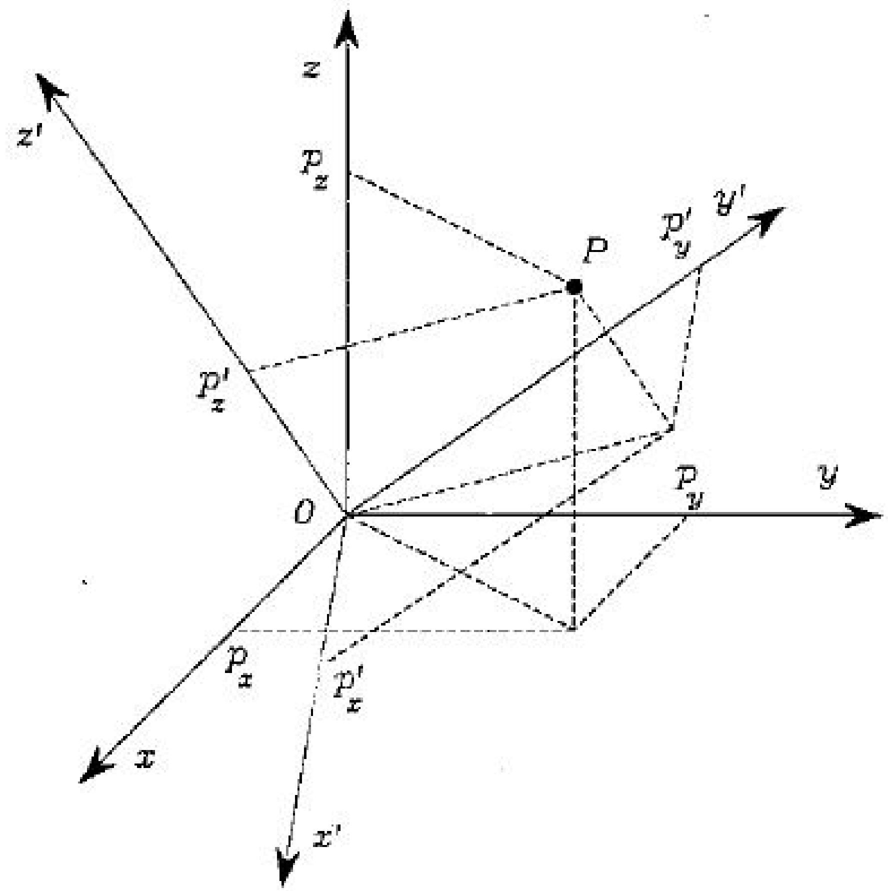


FIGURE 2.3
Representation of a point P in two different coordinate frames.

Composition of Rotational Matrices

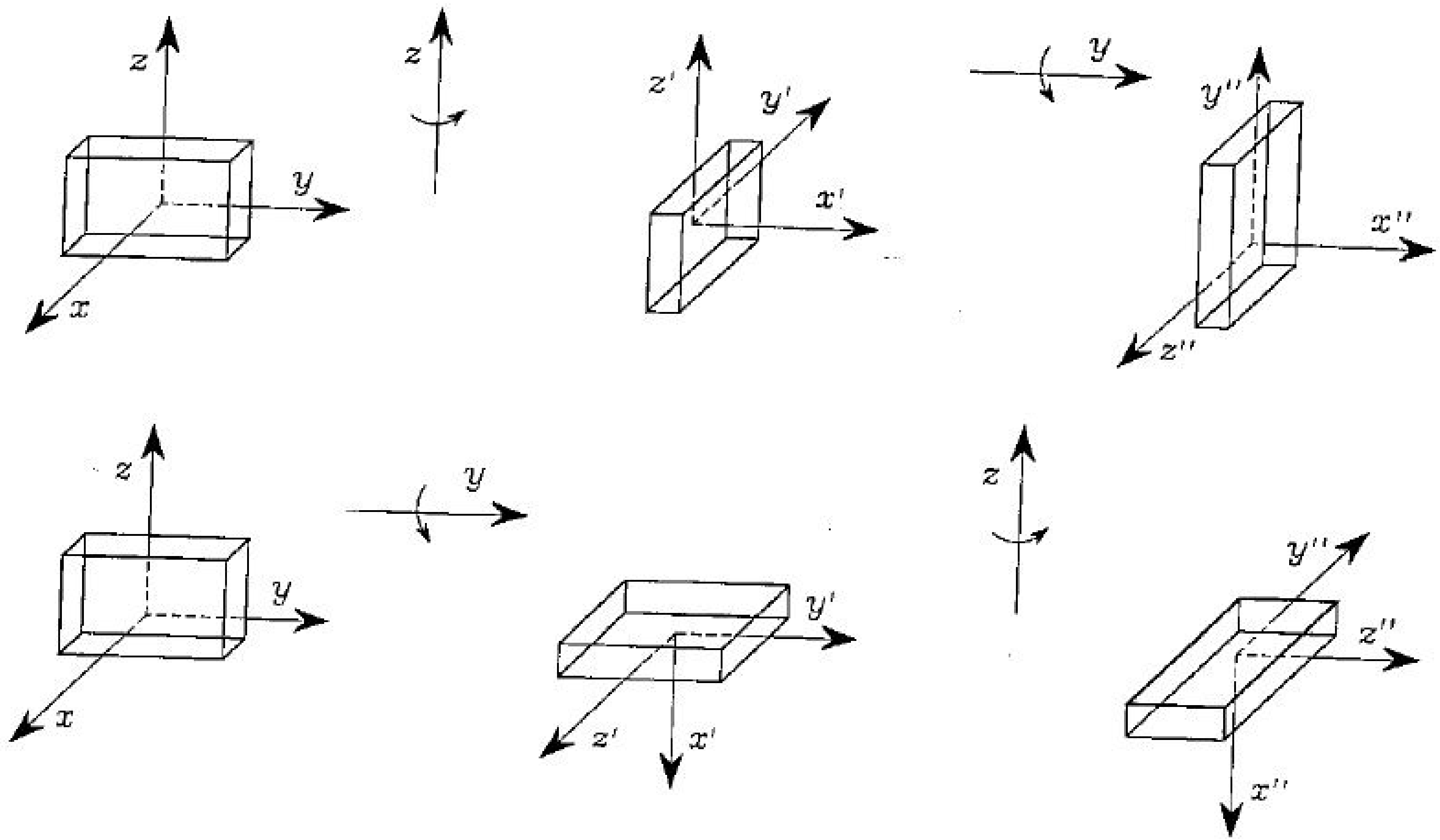
- Consider several frames
- R_i^j is transform from frame *i* to frame *j*

$$p^1 = R_2^1 p^2$$
$$p^0 = R_1^0 p^1$$
$$p^0 = R_2^0 p^2$$

- Can express rotation as series of partial rotations:

$$R_2^0 = R_1^0 R_2^1$$

Composition of Rotational Matrices



- Order matters!!!

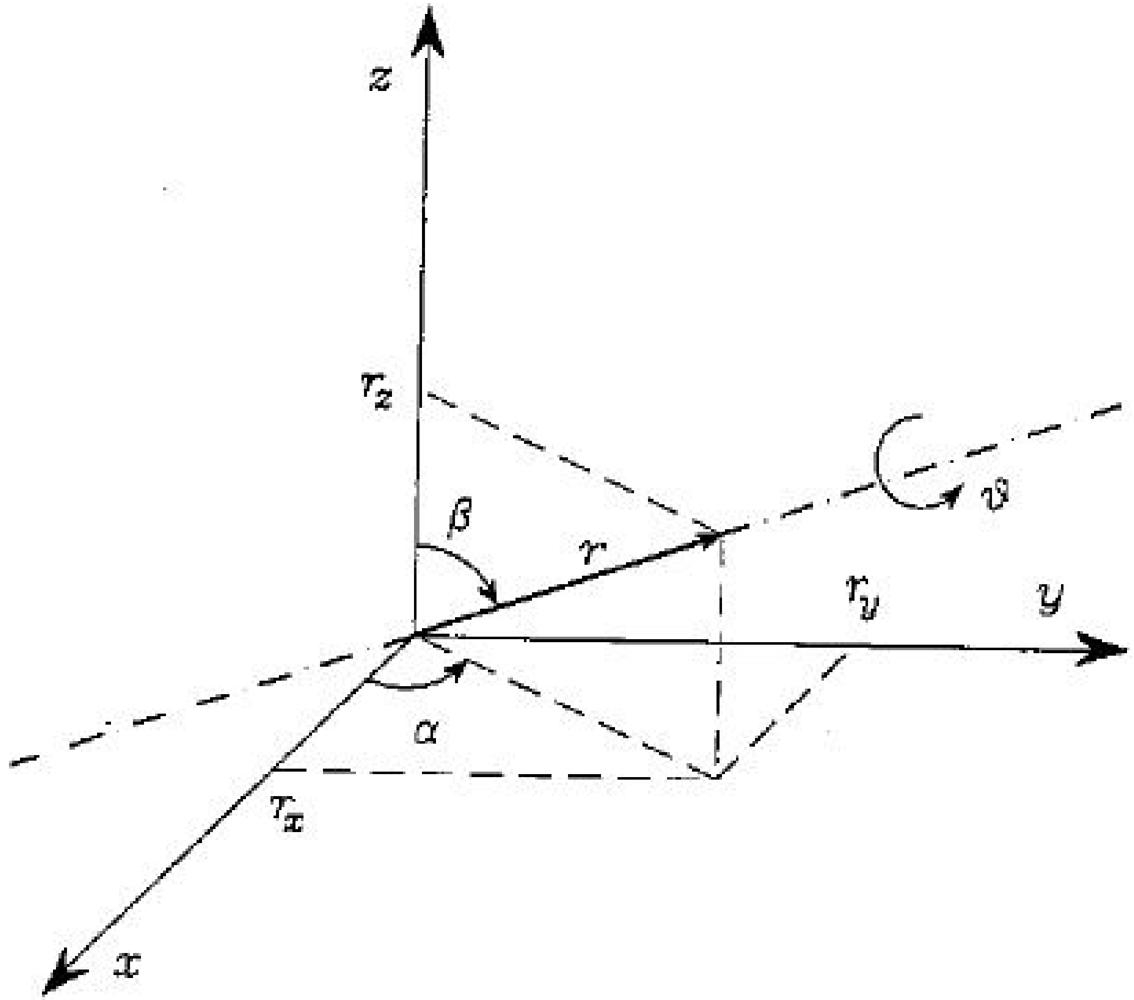
Rotation around Arbitrary Axis

- $\boldsymbol{r} = [r_x \quad r_y \quad r_z]^T$: unit vector
- Want to rotate by ϑ

1. Align *r* with *z* by rotating around *z* and *y*
2. Rotate ϑ around *z*
3. Realign with original direction

$$\boldsymbol{R}_r(\vartheta) = \boldsymbol{R}_z(\alpha)\boldsymbol{R}_y(\beta)\boldsymbol{R}_z(\vartheta)\boldsymbol{R}_y(-\beta)\boldsymbol{R}_z(-\alpha).$$

Rotation around Arbitrary Axis



Rotation around Arbitrary Axis

- Remove dependency on α and β:

$$\sin\alpha = \frac{r_y}{\sqrt{r_x^2 + r_y^2}} \qquad \cos\alpha = \frac{r_x}{\sqrt{r_x^2 + r_y^2}}$$

$$\sin\beta = \sqrt{r_x^2 + r_y^2} \qquad \cos\beta = r_z.$$

- Thus, final transformation:

$$\boldsymbol{R}_r(\vartheta) = \begin{bmatrix} r_x^2(1-c_\vartheta)+c_\vartheta & r_x r_y(1-c_\vartheta)-r_z s_\vartheta & r_x r_z(1-c_\vartheta)+r_y s_\vartheta \\ r_x r_y(1-c_\vartheta)+r_z s_\vartheta & r_y^2(1-c_\vartheta)+c_\vartheta & r_y r_z(1-c_\vartheta)-r_x s_\vartheta \\ r_x r_z(1-c_\vartheta)-r_y s_\vartheta & r_y r_z(1-c_\vartheta)+r_x s_\vartheta & r_z^2(1-c_\vartheta)+c_\vartheta \end{bmatrix}$$

Rotation around Arbitrary Axis

- Inverse: Given rotation matrix

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

Find

$$\vartheta = \cos^{-1}\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right)$$

$$\boldsymbol{r} = \frac{1}{2\sin\vartheta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix},$$

With constraint: $r_x^2 + r_y^2 + r_z^2 = 1$

Euler Angles

- Last representation had 9 variables, although 6 constraints due to orthogonality.
- Really only **3** free parameters
- Want a more minimal representation
- Use **Euler angles**
- Let $(\varphi, \vartheta, \psi)$ be a given set of Euler angles
  1. Rotate φ around *z* axis $\boldsymbol{R}_z(\varphi)$
  2. Rotate ϑ around *y'* axis $\boldsymbol{R}_{y'}(\vartheta)$
  3. Rotate φ around *z''* axis $\boldsymbol{R}_{z''}(\psi)$
- Example of ZYZ Euler Angles. 12 Combinations

Euler Angles

$$
\begin{aligned}
\boldsymbol{R}_{\mathrm{EUL}} &= \boldsymbol{R}_z(\varphi)\boldsymbol{R}_{y'}(\vartheta)\boldsymbol{R}_{z''}(\psi) \\
&= \begin{bmatrix} c_\varphi c_\vartheta c_\psi - s_\varphi s_\psi & -c_\varphi c_\vartheta s_\psi - s_\varphi c_\psi & c_\varphi s_\vartheta \\ s_\varphi c_\vartheta c_\psi + c_\varphi s_\psi & -s_\varphi c_\vartheta s_\psi + c_\varphi c_\psi & s_\varphi s_\vartheta \\ -s_\vartheta c_\psi & s_\vartheta s_\psi & c_\vartheta \end{bmatrix}
\end{aligned}
$$

Euler Angles

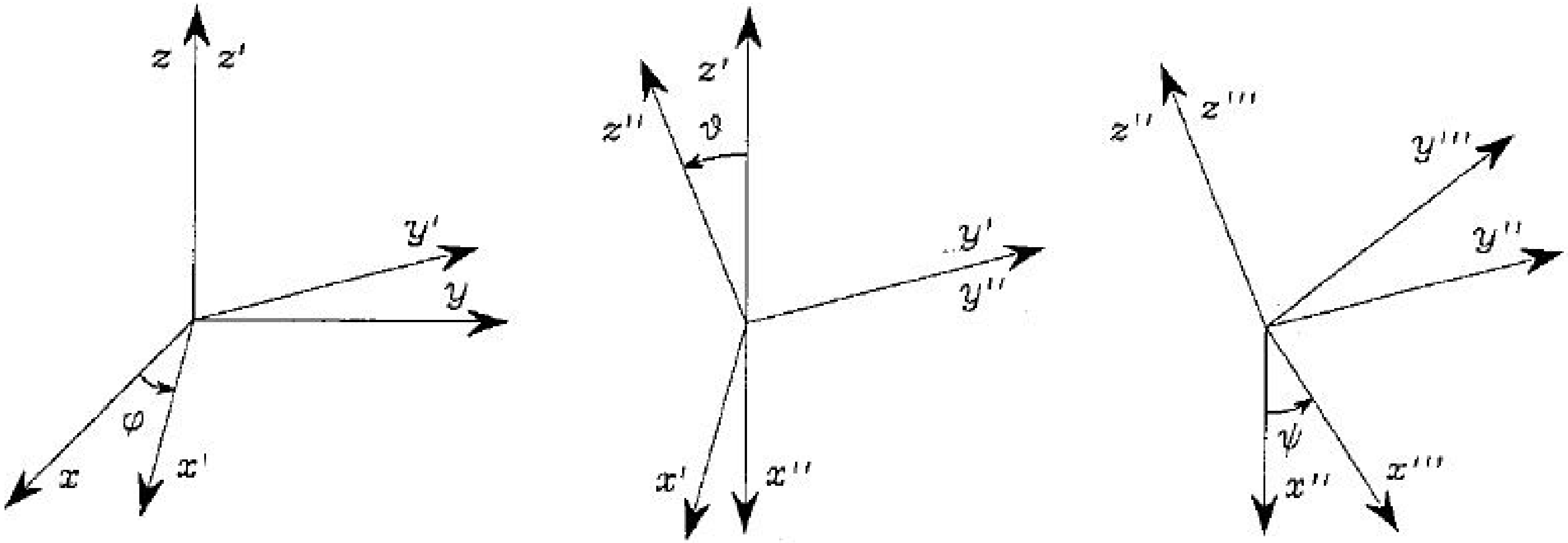


FIGURE 2.9
Representation of Euler angles ZYZ.

Homogeneous Transforms

- So far, only rotations. What about translations?

$$p^0 = o_1^0 + R_1^0 p^1$$

- Use *homogeneous transformation matrix*

$$\tilde{p} = \begin{bmatrix} p \\ 1 \end{bmatrix} \qquad A_1^0 = \begin{bmatrix} R_1^0 & o_1^0 \\ 0^T & 1 \end{bmatrix}$$

Homogeneous Transforms