# CS545—Contents IX

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- Reading Assignment for Next Class
    - See http://www-clmc.usc.edu/~cs545

# The Inverse Kinematics Problem

- Direct Kinematics

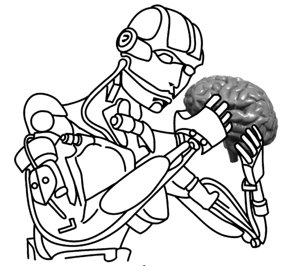
$$\mathbf{x} = f(\theta)$$

- Inverse Kinematics

$$\theta = f^{-1}(\mathbf{x})$$

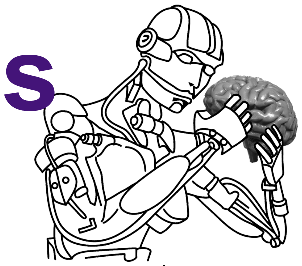
- Possible Problems of Inverse Kinematics
  - Multiple solutions
  - Infinitely many solutions
  - No solutions
  - No closed-form (analytical solution)

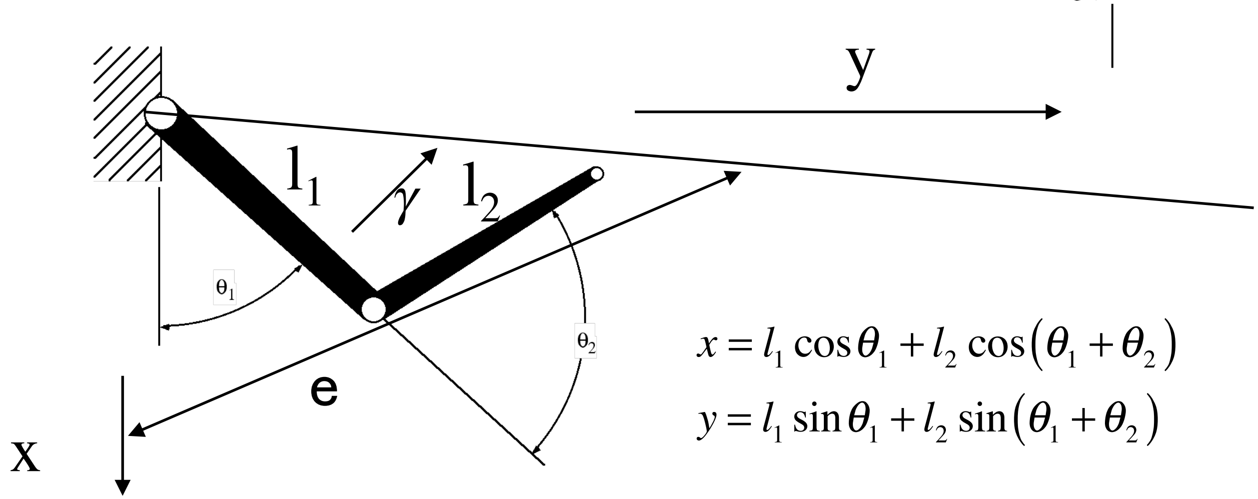
# Analytical (Algebraic) Solutions



- Analytically invert the direct kinematics equations and enumerate all solution branches
  - Note: this only works if the number of constraints is the same as the number of degrees-of-freedom of the robot
  - What if not?
    - Iterative solutions
    - Invent artificial constraints
- Examples
  - 2DOF arm
  - See S&S textbook 2.11 ff

# Analytical Inverse Kinematics of a 2 DOF Arm



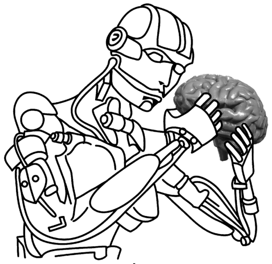



$$x = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2)$$
$$y = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2)$$

- Inverse Kinematics:

$$l = \sqrt{x^2 + y^2}$$
$$l_2^2 = l_1^2 + l^2 - 2l_1 l \cos\gamma$$
$$\Rightarrow \gamma = \arccos\left(\frac{l^2 + l_1^2 - l_2^2}{2l_1 l}\right)$$
$$\frac{y}{x} = \tan\varepsilon \quad \Rightarrow \quad \theta_1 = \arctan\frac{y}{x} - \gamma$$
$$\theta_2 = \arctan\left(\frac{y - l_1 \sin\theta}{x - l_1 \cos\theta_1}\right) - \theta_1$$

# Iterative Solutions of Inverse Kinematics



- Resolved Motion Rate Control

$$\dot{\mathbf{x}} = J(\theta)\dot{\theta} \quad \Rightarrow$$

$$\dot{\theta} = J(\theta)^{\#}\,\dot{\mathbf{x}}$$

- Properties
  - Only holds for high sampling rates or low Cartesian velocities
  - “a local solution” that may be “globally” inappropriate
  - Problems with singular postures
  - Can be used in two ways:
    - As an instantaneous solutions of “which way to take “
    - As an “batch” iteration method to find the correct configuration at a target

# Essential in Resolved Motion Rate Methods: The Jacobian

- Jacobian of direct kinematics:

$$\mathbf{x} = f(\theta) \quad \Rightarrow$$

$$\frac{\partial \mathbf{x}}{\partial \theta} = \frac{\partial f(\theta)}{\partial \theta} = J(\theta)$$

Analytical Jacobian

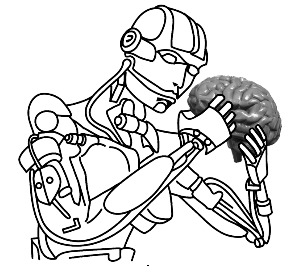
- In general, the Jacobian (for Cartesian positions and orientations) has the following form (geometrical Jacobian):

$$J(\theta) = \begin{pmatrix} j_{P1} & & j_{Pn} \\ & \ldots & \\ j_{O1} & & j_{On} \end{pmatrix}$$

$$\text{where} \quad \begin{bmatrix} j_{Pi} \\ j_{Oi} \end{bmatrix} = \begin{cases} \begin{bmatrix} z_{i-1} \\ \mathbf{0} \end{bmatrix} & \text{for a prismatic joint} \\ \begin{bmatrix} z_{i-1} \times (\mathbf{p} - \mathbf{p}_{i-1}) \\ z_{i-1} \end{bmatrix} & \text{for a revolute joint} \end{cases}$$

$\mathbf{p}_i$ is the vector from the origin of the world coordinate system to the origin of the i-th link coordinate system, p is the vector from the origin to the endeffector end, and **z** is the i-th joint axis (p.72 S&S)

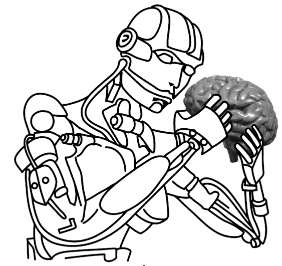
# The Jacobian Transpose Method



$$\Delta\theta = \alpha\, J^T(\theta)\, \Delta\mathbf{x}$$

- Operating Principle:
  - Project difference vector D**x** on those dimensions **q** which can reduce it the most
- Advantages:
  - Simple computation (numerically robust)
  - No matrix inversions
- Disadvantages:
  - Needs many iterations until convergence in certain configurations (e.g., Jacobian has very small coefficients)
  - Unpredictable joint configurations
  - Non conservative

# Jacobian Transpose Derivation



Minimize cost function $F = \frac{1}{2}\left(\mathbf{x}_{target} - \mathbf{x}\right)^T \left(\mathbf{x}_{target} - \mathbf{x}\right)$

$$= \frac{1}{2}\left(\mathbf{x}_{target} - f(\boldsymbol{\theta})\right)^T \left(\mathbf{x}_{target} - f(\boldsymbol{\theta})\right)$$

with respect to $\boldsymbol{\theta}$ by gradient descent:

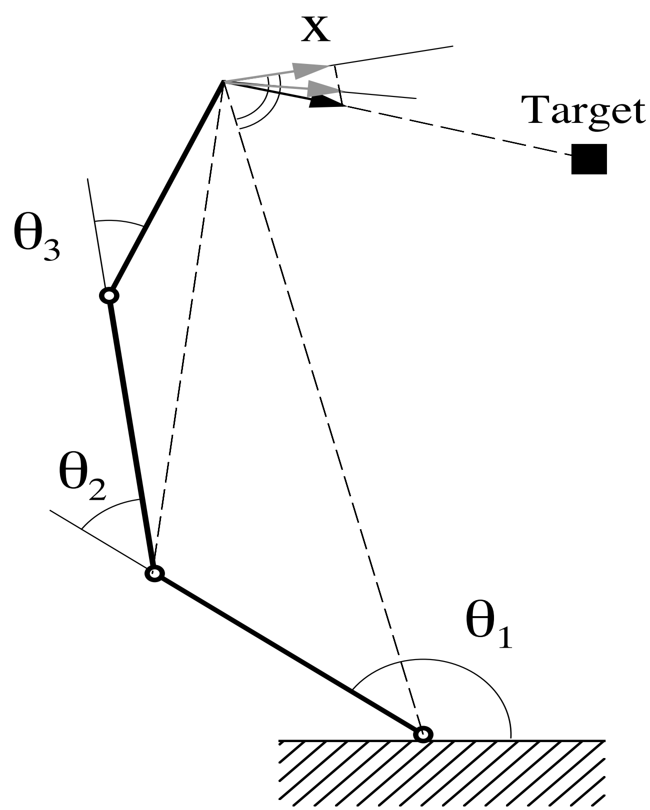
$$\Delta\boldsymbol{\theta} = -\alpha\left(\frac{\partial F}{\partial \boldsymbol{\theta}}\right)^T$$

$$= \alpha\left(\left(\mathbf{x}_{target} - \mathbf{x}\right)^T \frac{\partial f(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right)^T$$

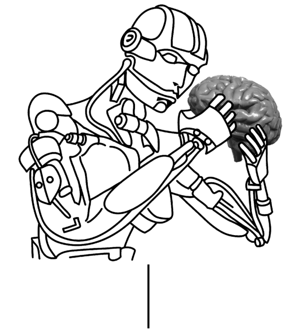
$$= \alpha\, J^T(\boldsymbol{\theta})\left(\mathbf{x}_{target} - \mathbf{x}\right)$$

$$= \alpha\, J^T(\boldsymbol{\theta})\Delta\mathbf{x}$$

# Jacobian Transpose
# Geometric Intuition



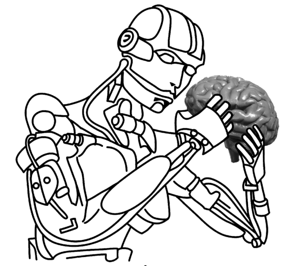
# The Pseudo Inverse Method



$$\Delta\boldsymbol{\theta} = \alpha\, J^T(\boldsymbol{\theta})\left(J(\boldsymbol{\theta})J^T(\boldsymbol{\theta})\right)^{-1}\Delta\mathbf{x} = J^{\#}\Delta\mathbf{x}$$

- Operating Principle:
  - Shortest path in **q**-space
- Advantages:
  - Computationally fast (second order method)
- Disadvantages:
  - Matrix inversion necessary (numerical problems)
  - Unpredictable joint configurations
  - Non conservative

# Pseudo Inverse Method Derivation



For a small step $\Delta\mathbf{x}$, minimize with repect to $\Delta\boldsymbol{\theta}$ the cost function:

$$F = \frac{1}{2}\Delta\boldsymbol{\theta}^T\Delta\boldsymbol{\theta} + \boldsymbol{\lambda}^T\left(\Delta\mathbf{x} - J(\boldsymbol{\theta})\Delta\boldsymbol{\theta}\right)$$

where $\boldsymbol{\lambda}^T$ is a vector of Lagrange multipliers.

Solution:

(1) $\frac{\partial F}{\partial \boldsymbol{\lambda}} = 0 \quad \Rightarrow \quad \Delta\mathbf{x} = J\Delta\boldsymbol{\theta}$

(2) $\frac{\partial F}{\partial \Delta\boldsymbol{\theta}} = 0 \quad \Rightarrow \quad \Delta\boldsymbol{\theta} = J^T\boldsymbol{\lambda} \quad \Rightarrow \quad J\Delta\boldsymbol{\theta} = JJ^T\boldsymbol{\lambda}$

$\Rightarrow \quad \boldsymbol{\lambda} = \left(JJ^T\right)^{-1}J\Delta\boldsymbol{\theta}$

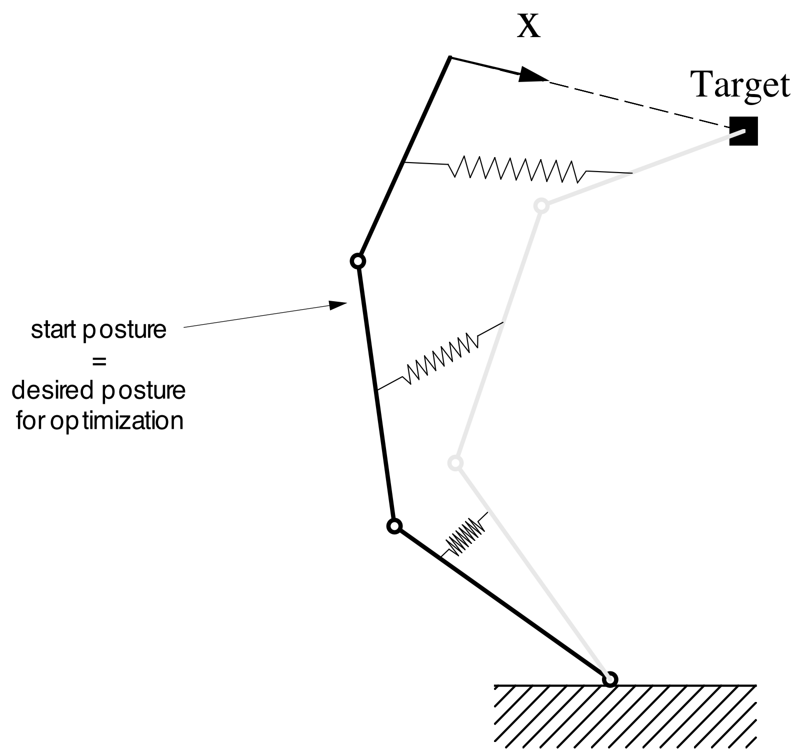
insert (1) into (2):

(3) $\boldsymbol{\lambda} = \left(JJ^T\right)^{-1}\Delta\mathbf{x}$

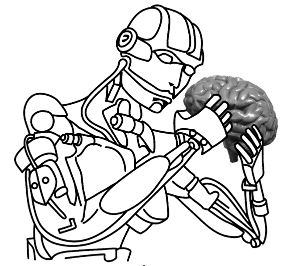
insertion of (3) into (2) gives the final result:

$$\Delta\boldsymbol{\theta} = J^T\boldsymbol{\lambda} = J^T\left(JJ^T\right)^{-1}\Delta\mathbf{x}$$

# Pseudo Inverse Geometric Intuition



# Pseudo Inverse with explicit Optimization Criterion



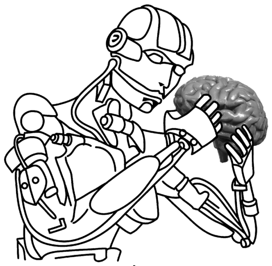
$$\Delta\boldsymbol{\theta} = \alpha J^{\#}\Delta\mathbf{x} + \left(I - J^{\#}J\right)\left(\boldsymbol{\theta}_O - \boldsymbol{\theta}\right)$$

- Operating Principle:
  - Optimization in null-space of Jacobian using a kinematic cost function

$$F = g(\boldsymbol{\theta}), \quad e.g., F = \sum_{i=1}^{d}\left(\theta_i - \theta_{i,0}\right)^2$$

- Advantages:
  - Computationally fast
  - Explicit optimization criterion provides control over arm configurations
- Disadvantages:
  - Numerical problems at singularities
  - Non conservative

# Pseudo Inverse Method & Optimization Derivation



For a small step $\Delta\mathbf{x}$, minimize with repect to $\Delta\boldsymbol{\theta}$ the cost function:

$$F = \frac{1}{2}\left(\Delta\boldsymbol{\theta} + \boldsymbol{\theta} - \boldsymbol{\theta}_o\right)^T \left(\Delta\boldsymbol{\theta} + \boldsymbol{\theta} - \boldsymbol{\theta}_o\right) + \boldsymbol{\lambda}^T \left(\Delta\mathbf{x} - J(\boldsymbol{\theta})\Delta\boldsymbol{\theta}\right)$$

where $\boldsymbol{\lambda}^T$ is a vector of Lagrange multipliers.

Solution:

$$(1) \quad \frac{\partial F}{\partial \boldsymbol{\lambda}} = 0 \quad \Rightarrow \quad \Delta\mathbf{x} = J\Delta\boldsymbol{\theta}$$

$$(2) \quad \frac{\partial F}{\partial \Delta\boldsymbol{\theta}} = 0 \quad \Rightarrow \quad \Delta\boldsymbol{\theta} = J^T\boldsymbol{\lambda} - \left(\boldsymbol{\theta} - \boldsymbol{\theta}_o\right) \quad \Rightarrow \quad J\Delta\boldsymbol{\theta} = JJ^T\boldsymbol{\lambda} - J\left(\boldsymbol{\theta} - \boldsymbol{\theta}_o\right)$$

$$\Rightarrow \quad \boldsymbol{\lambda} = \left(JJ^T\right)^{-1} J\Delta\boldsymbol{\theta} + \left(JJ^T\right)^{-1} J\left(\boldsymbol{\theta} - \boldsymbol{\theta}_o\right)$$

insert (1) into (2):

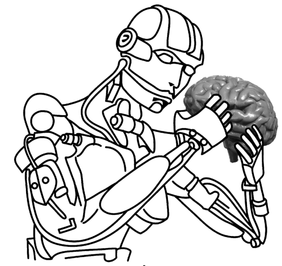
$$(3) \quad \boldsymbol{\lambda} = \left(JJ^T\right)^{-1} \Delta\mathbf{x} + \left(JJ^T\right)^{-1} J\left(\boldsymbol{\theta} - \boldsymbol{\theta}_o\right)$$

insertion of (3) into (2) gives the final result:

$$\Delta\boldsymbol{\theta} = J^T\boldsymbol{\lambda} - \left(\boldsymbol{\theta} - \boldsymbol{\theta}_o\right) = J^T\left(JJ^T\right)^{-1}\Delta\mathbf{x} + J^T\left(JJ^T\right)^{-1} J\left(\boldsymbol{\theta} - \boldsymbol{\theta}_o\right) - \left(\boldsymbol{\theta} - \boldsymbol{\theta}_o\right)$$

$$= J^{\#}\Delta\mathbf{x} + \left(I - J^{\#}J\right)\left(\boldsymbol{\theta}_o - \boldsymbol{\theta}\right)$$

# The Extended Jacobian Method



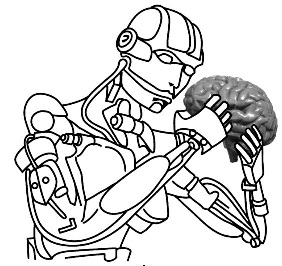
$$\Delta\boldsymbol{\theta} = \alpha\left(J^{ext.}(\boldsymbol{\theta})\right)^{-1}\Delta\mathbf{x}^{ext.}$$

- Operating Principle:
  - Optimization in null-space of Jacobian using a kinematic cost function

$$F = g(\boldsymbol{\theta}), \quad e.g., F = \sum_{i=1}^{d}\left(\theta_i - \theta_{i,0}\right)^2$$

- Advantages:
  - Computationally fast (second order method)
  - Explicit optimization criterion provides control over arm configurations
  - Numerically robust
  - Conservative
- Disadvantages:
  - Computationally expensive matrix inversion necessary (singular value decomposition)
  - Note: new and better ext. Jac. algorithms exist

# Extended Jacobian Method Derivation



The forward kinematics $\mathbf{x} = f(\boldsymbol{\theta})$ is a mapping $\Re^n \rightarrow \Re^m$, e.g., from a $n$-dimensional joint space to a $m$-dimensional Cartesian space. The singular value decomposition of the Jacobian of this mapping is:

$$J(\boldsymbol{\theta}) = \mathbf{USV}^T$$

The rows $[\mathbf{V}]_i$ whose corresponding entry in the diagonal matrix $\mathbf{S}$ is zero are the vectors which span the Null space of $J(\boldsymbol{\theta})$. There must be (at least) $n$-$m$ such vectors ($n \geq m$). Denote these vectors $\mathbf{n}_i, i \in [1, n-m]$. The goal of the extended Jacobian method is to augment the rank deficient Jacobian such that it becomes properly invertible. In order to do this, a cost function $F$=$g(\boldsymbol{\theta})$ has to be defined which is to be minimized with respect to $\boldsymbol{\theta}$ in the Null space. Minimization of F must always yield:

$$\frac{\partial F}{\partial \boldsymbol{\theta}} = \frac{\partial g}{\partial \boldsymbol{\theta}} = 0$$

Since we are only interested in zeroing the gradient in Null space, we project this gradient onto the Null space basis vectors:

$$G_i = \frac{\partial g}{\partial \boldsymbol{\theta}} \mathbf{n}_i$$

If all $G_i$ equal zero, the cost function F is minimized in Null space. Thus we obtain the following set of equations which are to be fulfilled by the inverse kinematics solution:

$$\begin{pmatrix} f(\boldsymbol{\theta}) \\ G_1 \\ \ldots \\ G_{n-m} \end{pmatrix} = \begin{pmatrix} \mathbf{x} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

For an incremental step $\Delta\mathbf{x}$, this system can be linearized:

$$\begin{pmatrix} J(\boldsymbol{\theta}) \\ \frac{\partial G_1}{\partial \boldsymbol{\theta}} \\ \ldots \\ \frac{\partial G_{n-m}}{\partial \boldsymbol{\theta}} \end{pmatrix} \Delta\boldsymbol{\theta} = \begin{pmatrix} \Delta\mathbf{x} \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad or \quad J^{ext.} \Delta\boldsymbol{\theta} = \Delta\mathbf{x}^{ext.}.$$

The unique solution of these equations is: $\Delta\boldsymbol{\theta} = \left(J^{ext.}\right)^{-1} \Delta\mathbf{x}^{ext.}$.

# Extended Jacobian Geometric Intuition