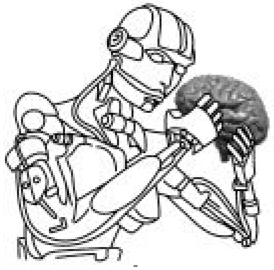
CS 545 – Lecture 8
Direct Kinematics

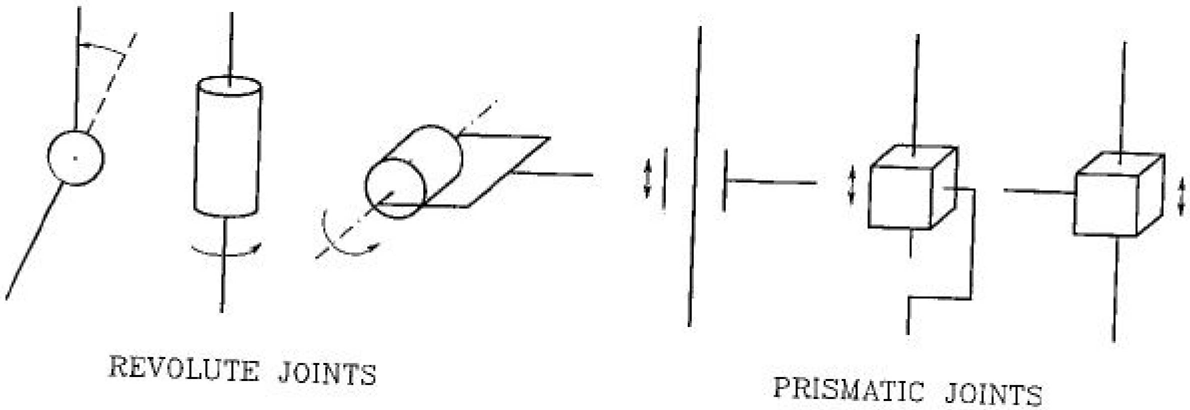
- Transformation from Joint Space to End-effector Space
- Denavit-Hartenberg Convention
- Examples
- Workspace Consideration

http://robotics.usc.edu/~aatrash/cs545

Direct Kinematics

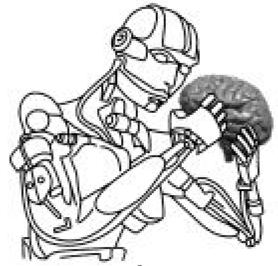


- Manipulator has *links* and *joints*
- *Revolute* and *prismatic* joints

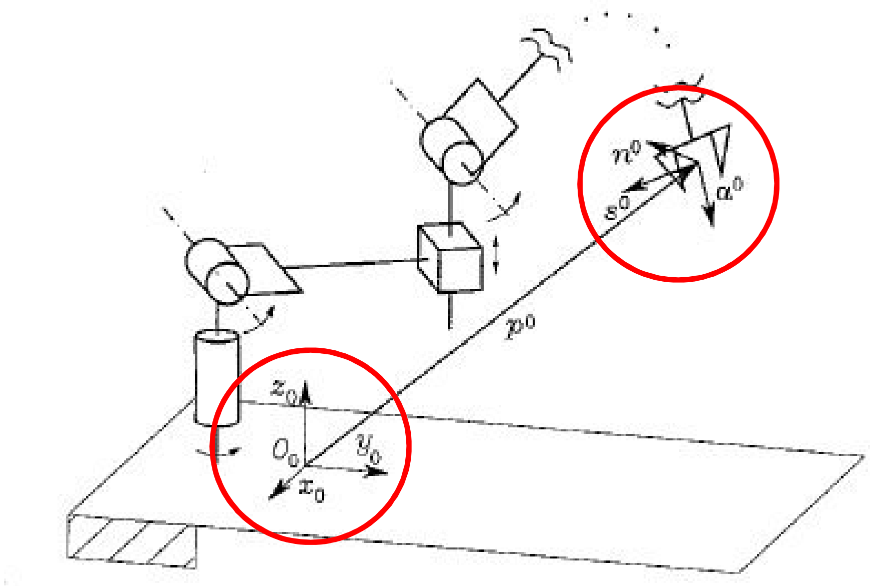


- *Base* and *endeffector*
- Entire structure: *open kinematic chain*

Direct Kinematics



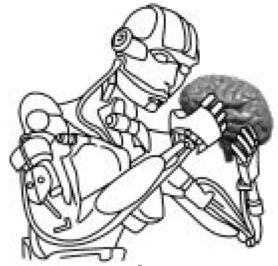
- *Direct kinematics* – Determine the end effector position and orientation as a function of joint variables



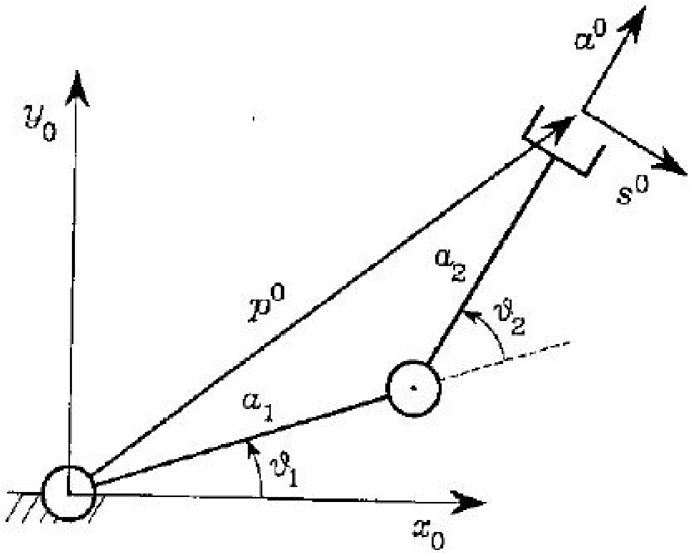
- *End-effector frame*
- *Base frame*

$$T^0(q) = \begin{bmatrix} n^0(q) & s^0(q) & a^0(q) & p^0(q) \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

Direct Kinematics



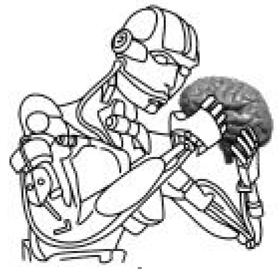
- Geometric analysis



$s_{ij} = \sin(q_i + q_j)$
$c_{ij} = \cos(q_i + q_j)$

$$T^0(q) = \begin{bmatrix} n^0 & s^0 & a^0 & p^0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & s_{12} & c_{12} & a_1c_1 + a_2c_{12} \\ 0 & -c_{12} & s_{12} & a_1s_1 + a_2s_{12} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

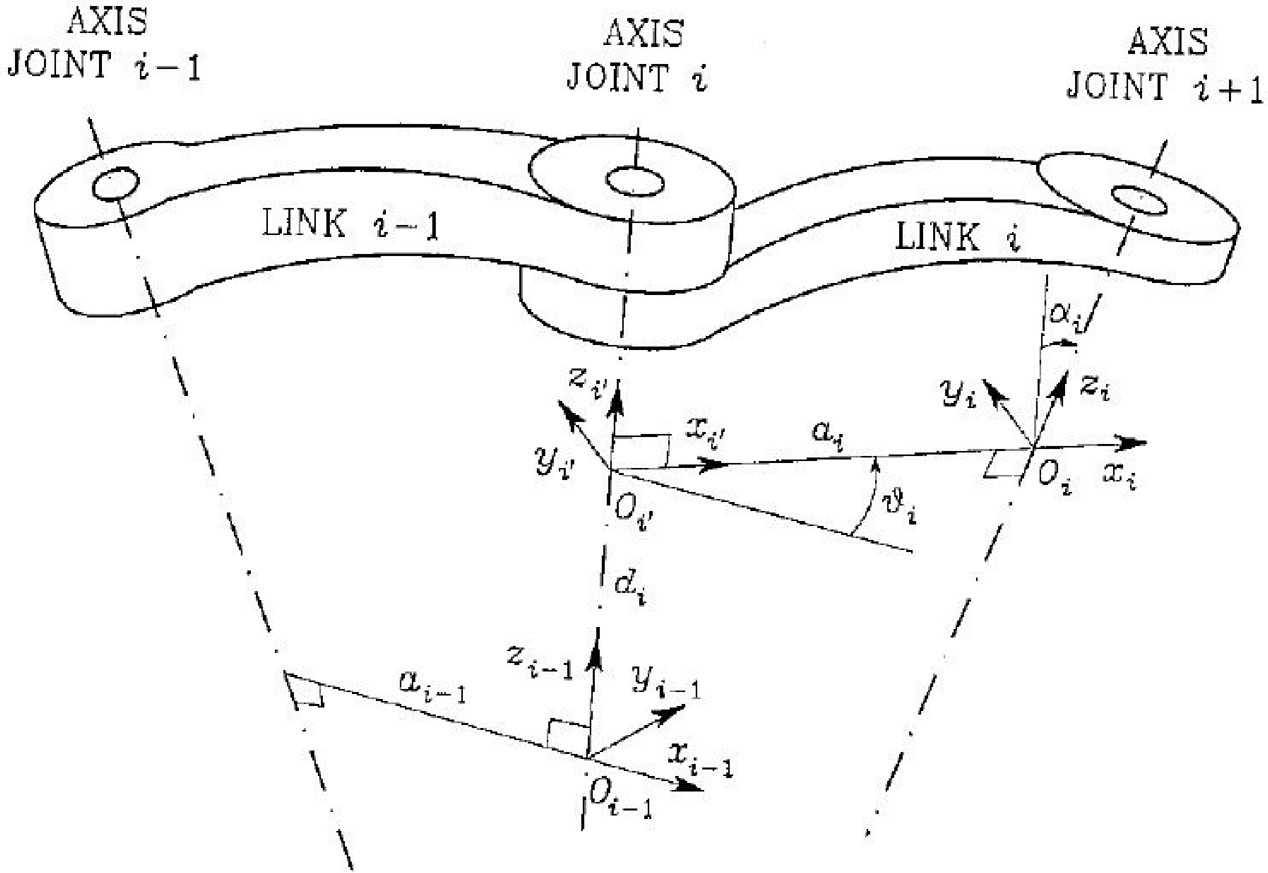
Denavit-Hartenberg Convention



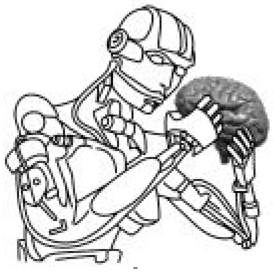
- Assume each joint connects two and only two consecutive links
- Describe this relationship and solve overall description recursively
- Use a set of rules (conventions really) known as *Denavit-Hartenberg Convention*
- (also called *DH Frames*)

- **Goal:** Find transform from link *i* to link *i+1*

Denavit-Hartenberg Convention

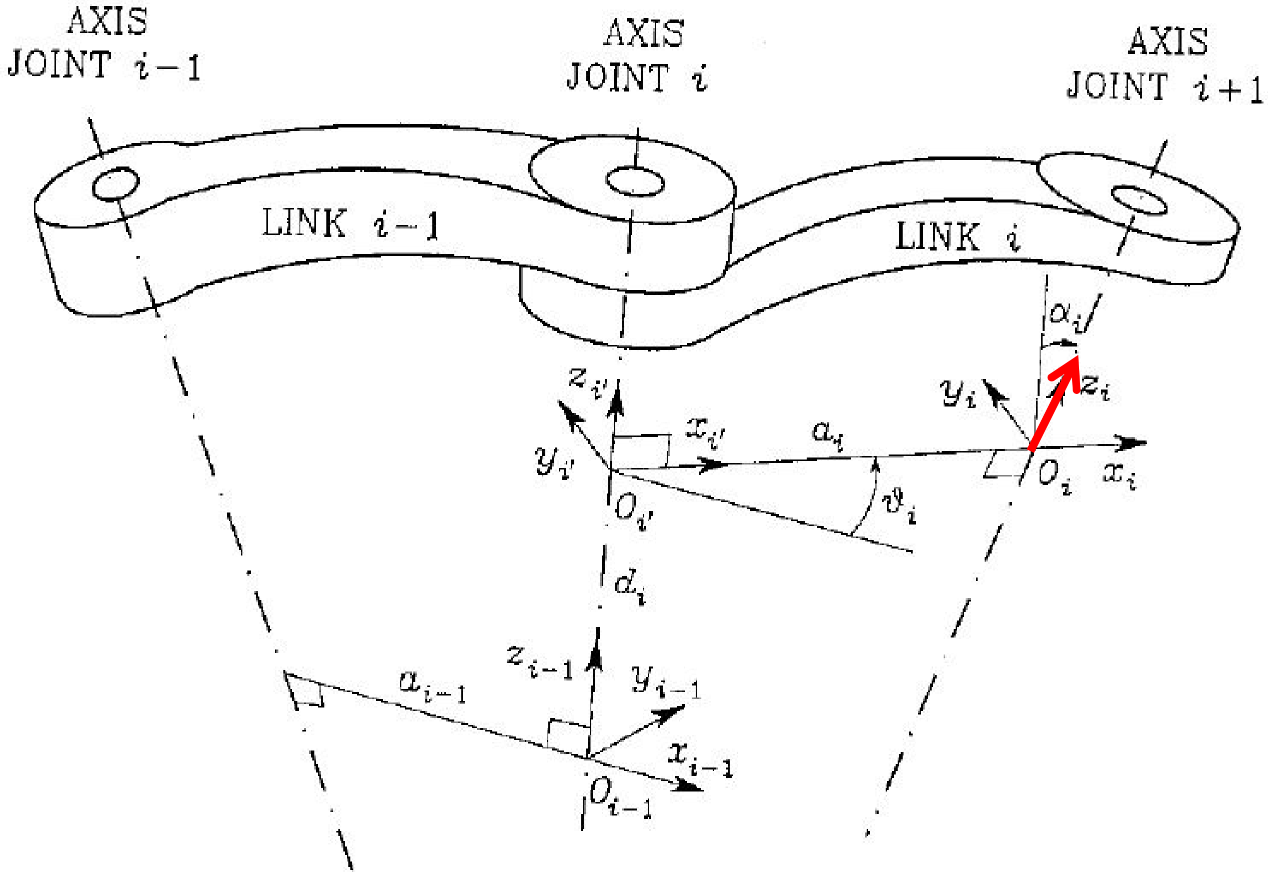


Denavit-Hartenberg Convention

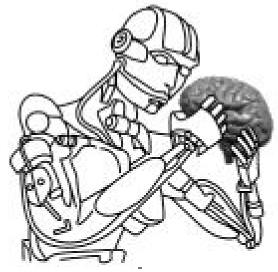


- Choose axis z_i along axis of joint $i+1$
- Location origin O_i at intersection of z_i and common normal with z_{i-1}.
- Location origin $O_{i'}$ at intersection of z_{i-1} and common normal with z_i.
- Choose axis x_i along common normal to axes z_{i-1} and z_i
- Choose axis y_i to complete right-handed frame

Denavit-Hartenberg Convention



Denavit-Hartenberg Convention



- Choose axis z_i along axis of joint $i+1$

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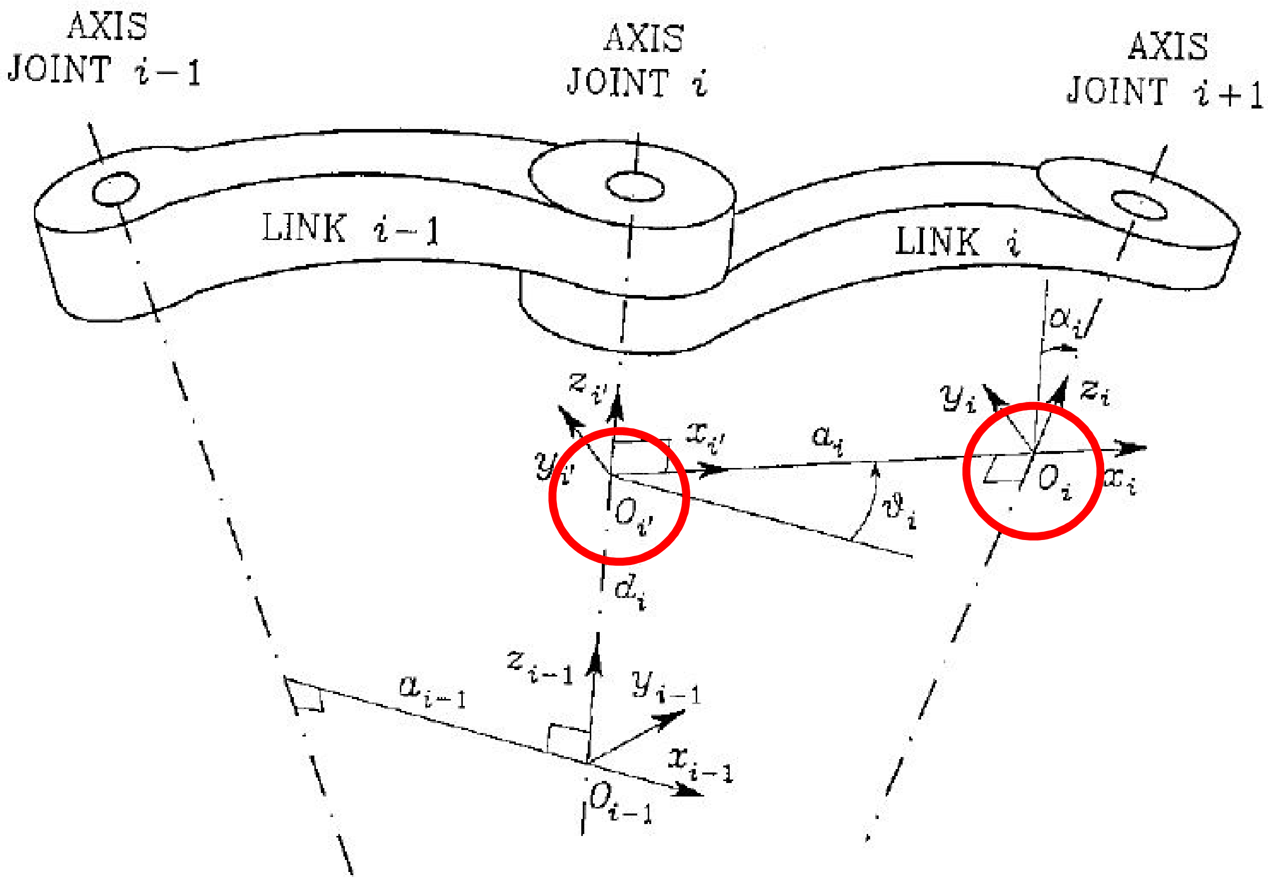
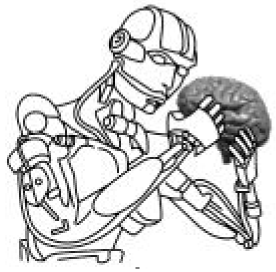
Denavit-Hartenberg Convention



Denavit-Hartenberg Convention



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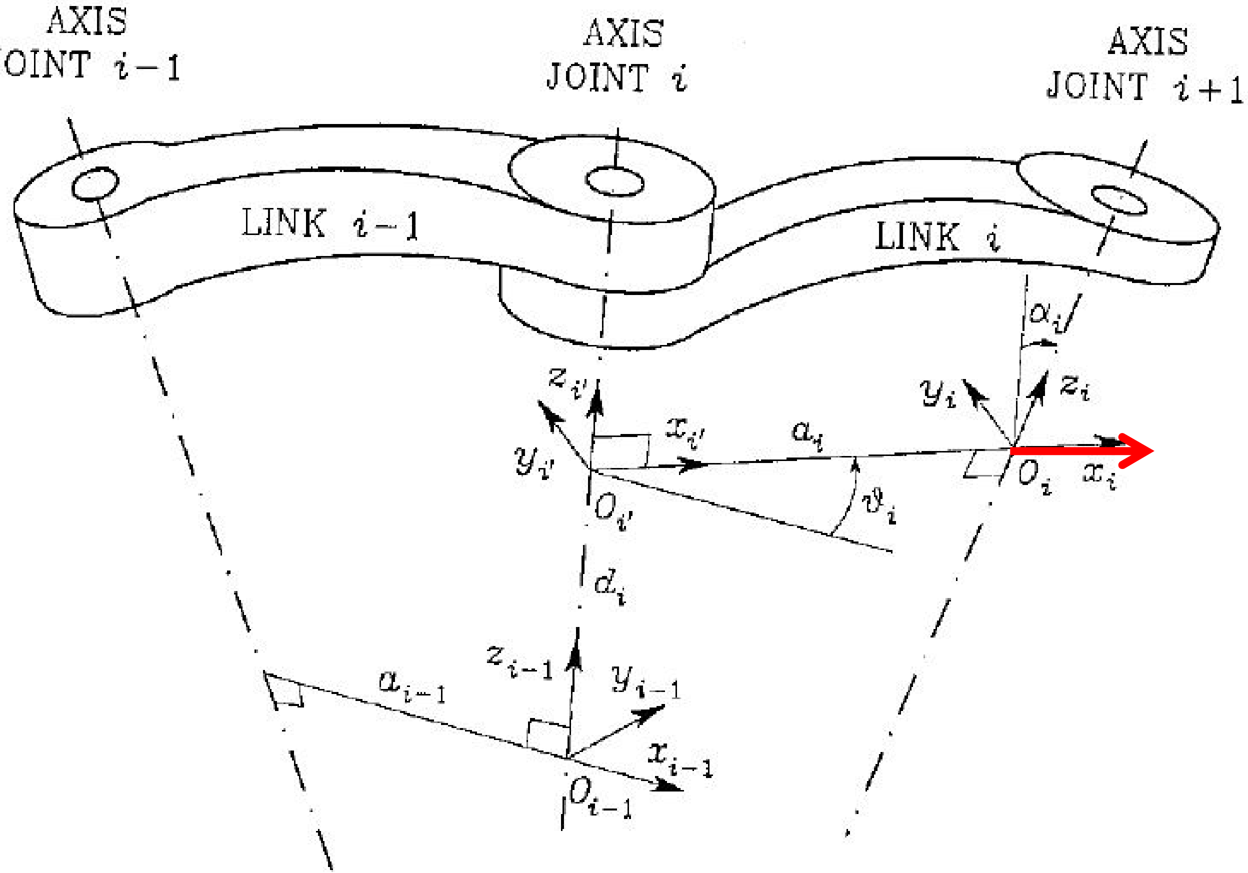

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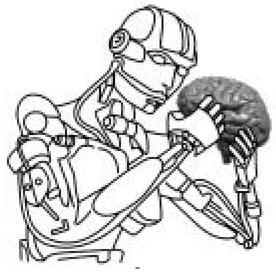

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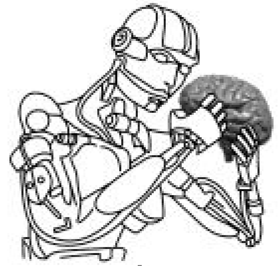
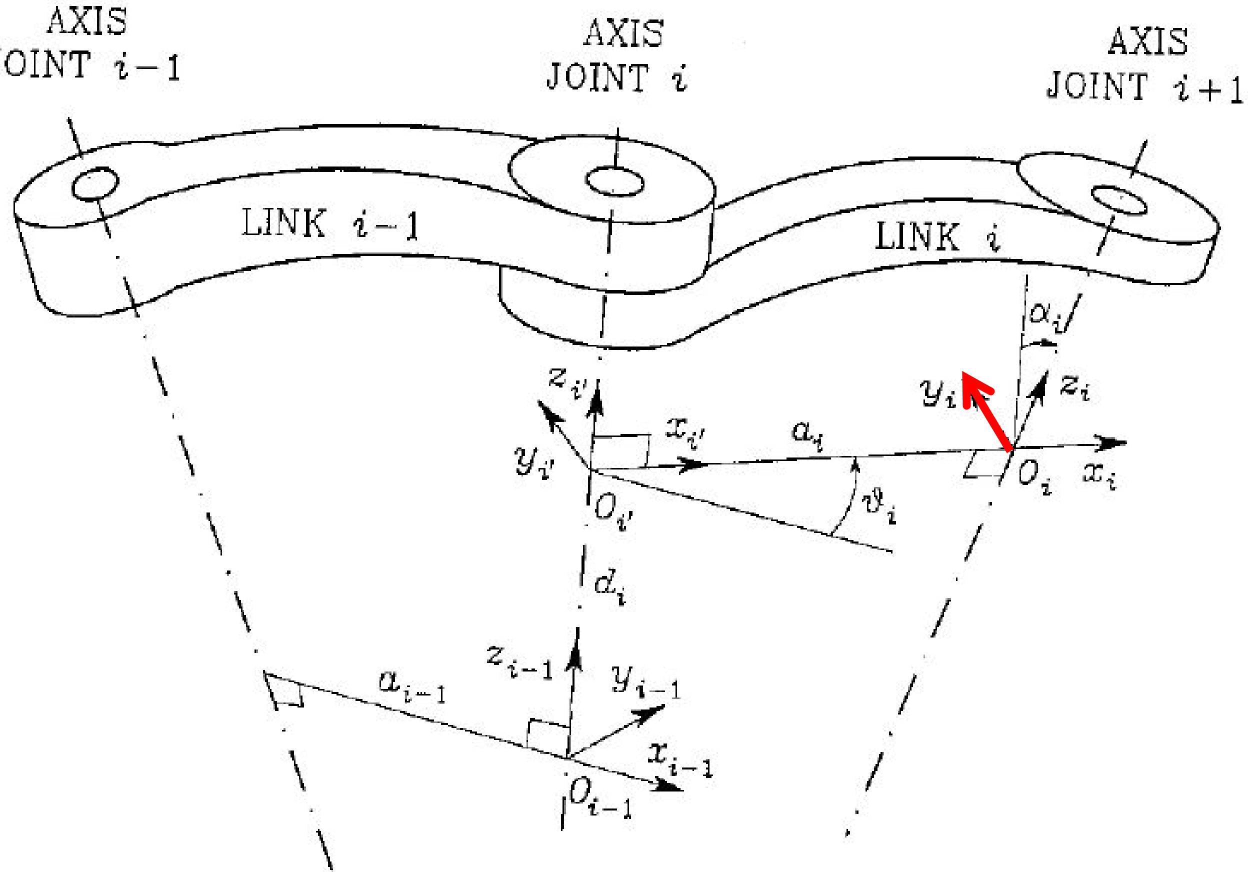
Denavit-Hartenberg Convention





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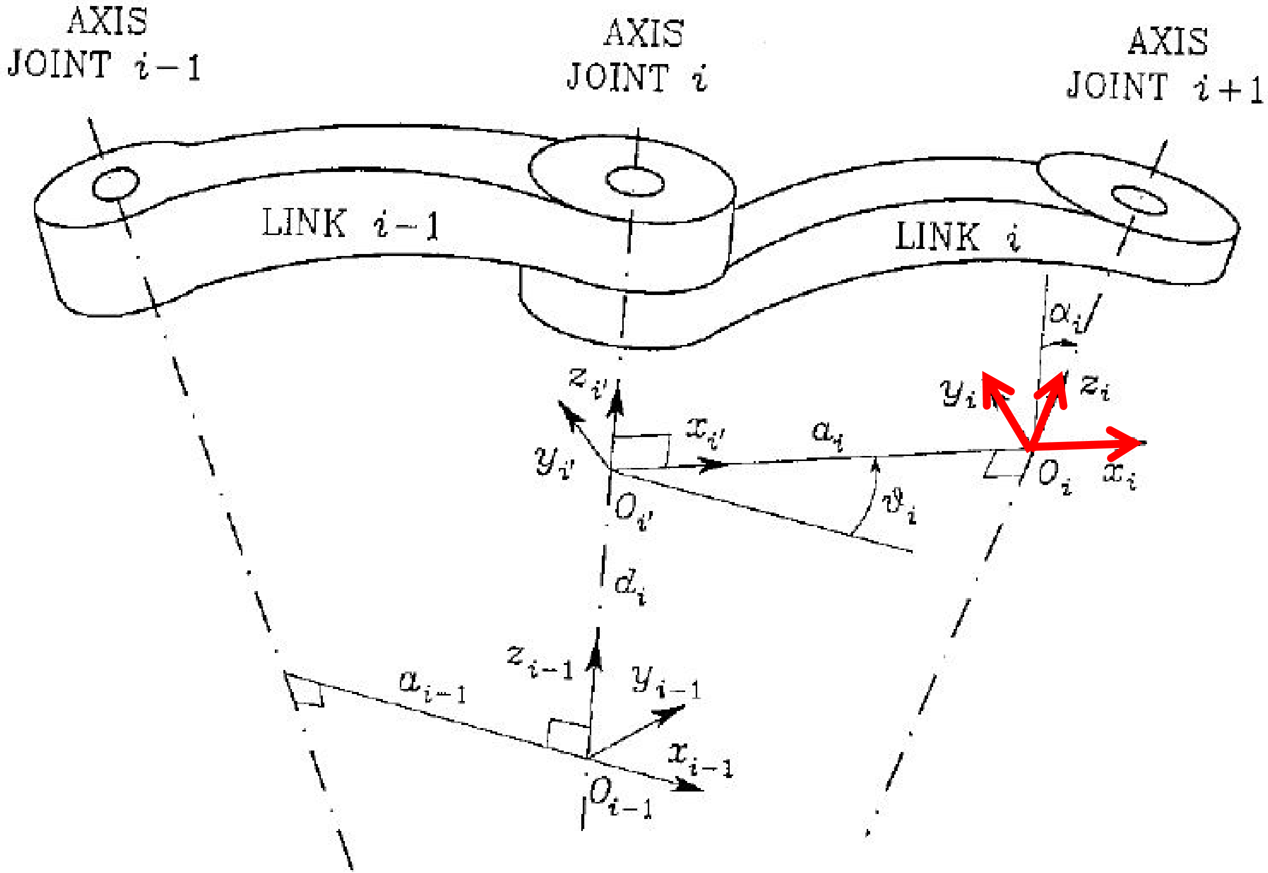

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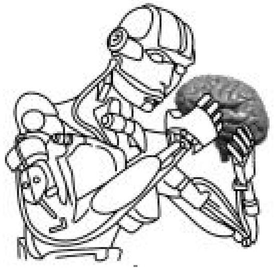


- Relationship beween frame *i-1* and frame *i* defined by 4 parameters:

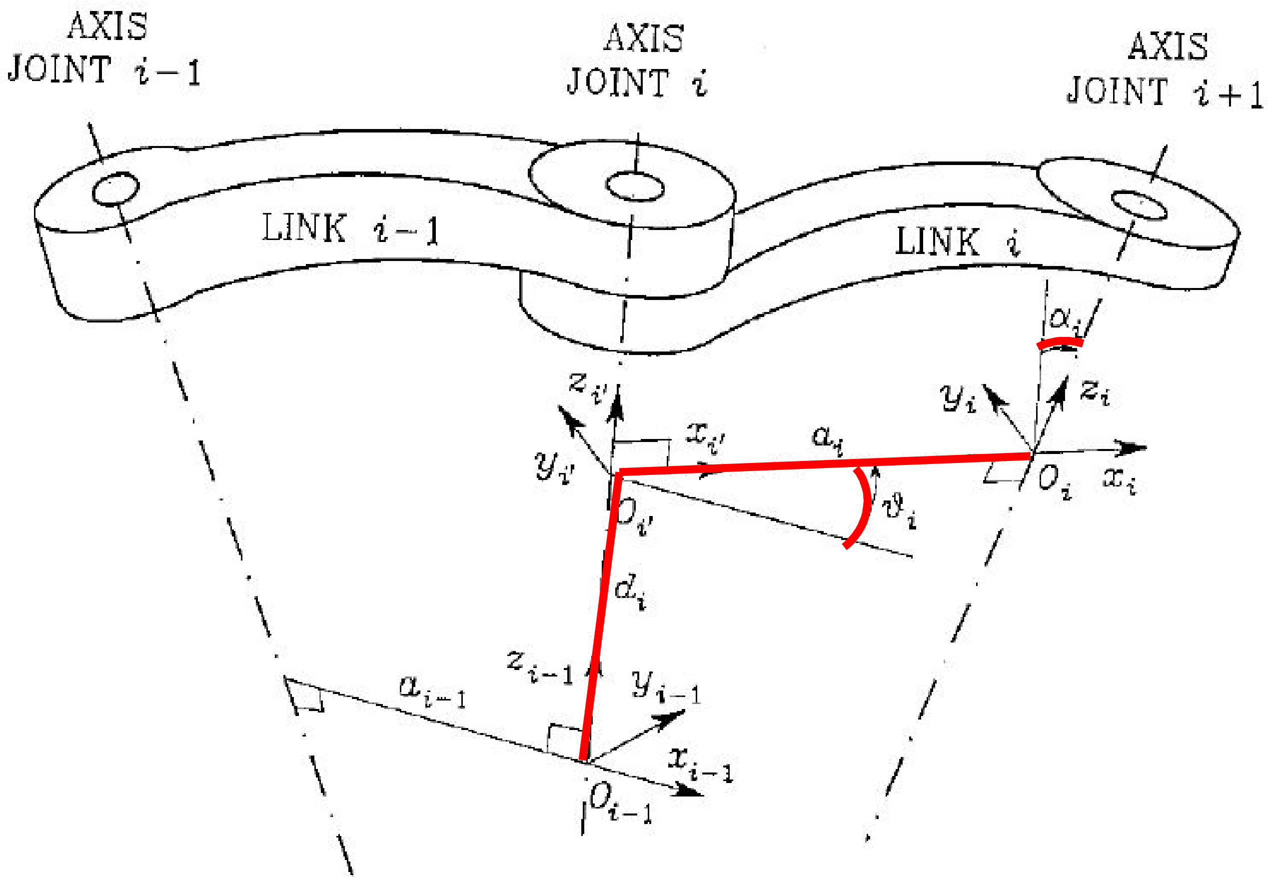
a_i distance between O_i and $O_{i'}$,

d_i coordinate of $O_{i'}$ along z_{i-1},

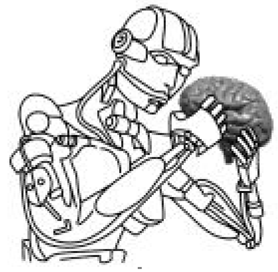
α_i angle between axes z_{i-1} and z_i about axis x_i to be taken positive when rotation is made counter-clockwise,

ϑ_i angle between axes x_{i-1} and x_i about axis z_{i-1} to be taken positive when rotation is made counter-clockwise.

Denavit-Hartenberg Convention

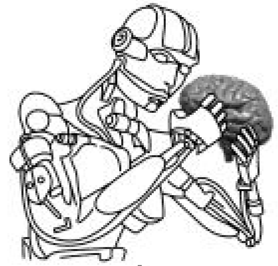


Denavit-Hartenberg Convention



- Parameters:
  - a_i – length of the link
  - d_i – displacement along *z* axis (how far from the same plane)
    - Only applies to *prismatic* joints
  - α_i – rotation along *z* axis
  - θ_i – rotation along *x* axis
    - only applies to *revolute* joints

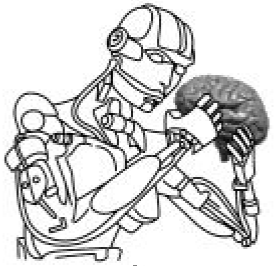
Denavit-Hartenberg Convention



$$\boldsymbol{A}_{i'}^{i-1} = \begin{bmatrix} c_{\vartheta_i} & -s_{\vartheta_i} & 0 & 0 \\ s_{\vartheta_i} & c_{\vartheta_i} & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{A}_{i}^{i'} = \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{A}_{i}^{i-1}(q_i) = \boldsymbol{A}_{i'}^{i-1}\boldsymbol{A}_{i}^{i'} = \begin{bmatrix} c_{\vartheta_i} & -s_{\vartheta_i}c_{\alpha_i} & s_{\vartheta_i}s_{\alpha_i} & a_i c_{\vartheta_i} \\ s_{\vartheta_i} & c_{\vartheta_i}c_{\alpha_i} & -c_{\vartheta_i}s_{\alpha_i} & a_i s_{\vartheta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Denavit-Hartenberg Convention

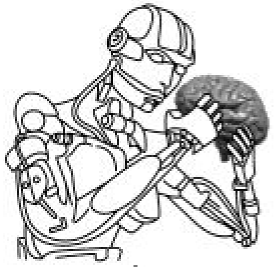


1. Find and number consecutively the joint axes; set the directions of axes $z_0, \ldots, z_{n-1}$.
2. Choose the base frame by locating the origin on axis z_0; axes x_0 and y_0 are chosen so as to obtain a right-handed frame.

Execute steps from **3** to **5** for $i = 1, \ldots, n-1$:

3. Locate the origin O_i at the intersection of z_i with the common normal to axes z_{i-1} and z_i. If axes z_{i-1} and z_i are parallel and joint i is revolute, then locate O_i so that $d_i = 0$; if joint i is prismatic, locate O_i at a reference position for the joint range, e.g., a mechanical limit.
4. Choose axis x_i along the common normal to axes z_{i-1} and z_i with direction from joint i to joint $i+1$.
5. Choose axis y_i so as to obtain a right-handed frame.

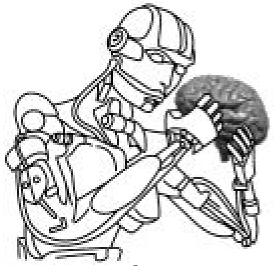
Denavit-Hartenberg Convention

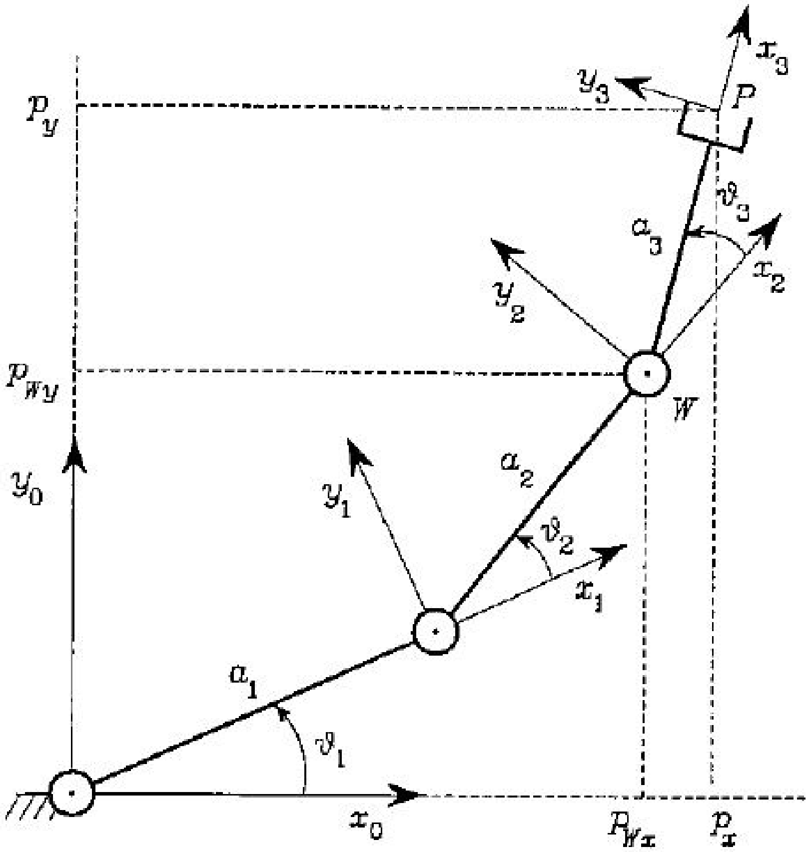


To complete:

6. Choose frame n with axis x_n normal to axis z_{n-1}; if joint n is revolute, then align z_n with the direction of z_{n-1}.
7. For $i = 1, \ldots, n$, form the table of parameters $a_i, d_i, \alpha_i, \vartheta_i$.
8. On the basis of the parameters in **7**, compute the homogeneous transformation matrices $\boldsymbol{A}_i^{i-1}(q_i)$ for $i = 1, \ldots, n$.
9. Compute the direct kinematics function $\boldsymbol{T}_n^0(\boldsymbol{q}) = \boldsymbol{A}_1^0 \ldots \boldsymbol{A}_n^{n-1}$ that yields the position and orientation of frame n with respect to the base frame.

Examples:
Three-Link Planar Arm



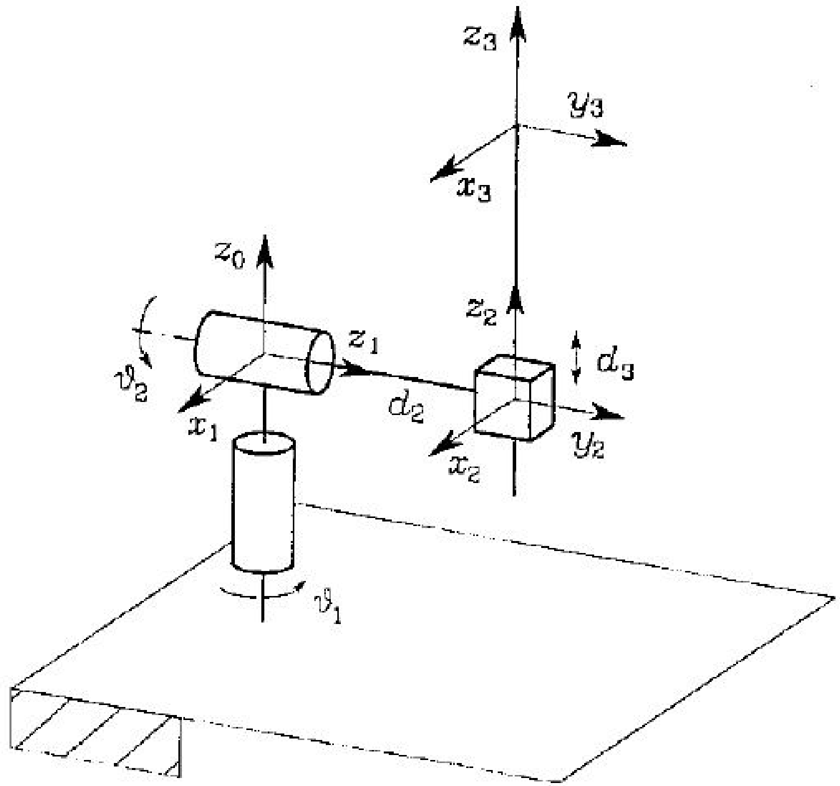


Link	a_i	α_i	d_i	ϑ_i
1	a_1	0	0	ϑ_1
2	a_2	0	0	ϑ_2
3	a_3	0	0	ϑ_3

$$A_i^{i-1}(\vartheta_i) = \begin{bmatrix} c_i & -s_i & 0 & a_i c_i \\ s_i & c_i & 0 & a_i s_i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad i = 1, 2, 3.$$

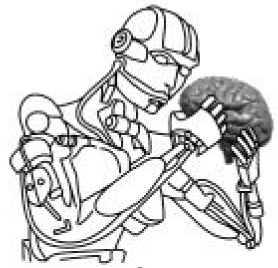
$$T_3^0(q) = A_1^0 A_2^1 A_3^2 = \begin{bmatrix} c_{123} & -s_{123} & 0 & a_1 c_1 + a_2 c_{12} + a_3 c_{123} \\ s_{123} & c_{123} & 0 & a_1 s_1 + a_2 s_{12} + a_3 s_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Examples:
Spherical Arm



Link	a_i	α_i	d_i	ϑ_i
1	0	$-\pi/2$	0	ϑ_1
2	0	$\pi/2$	d_2	ϑ_2
3	0	0	d_3	0

Examples:
Spherical Arm

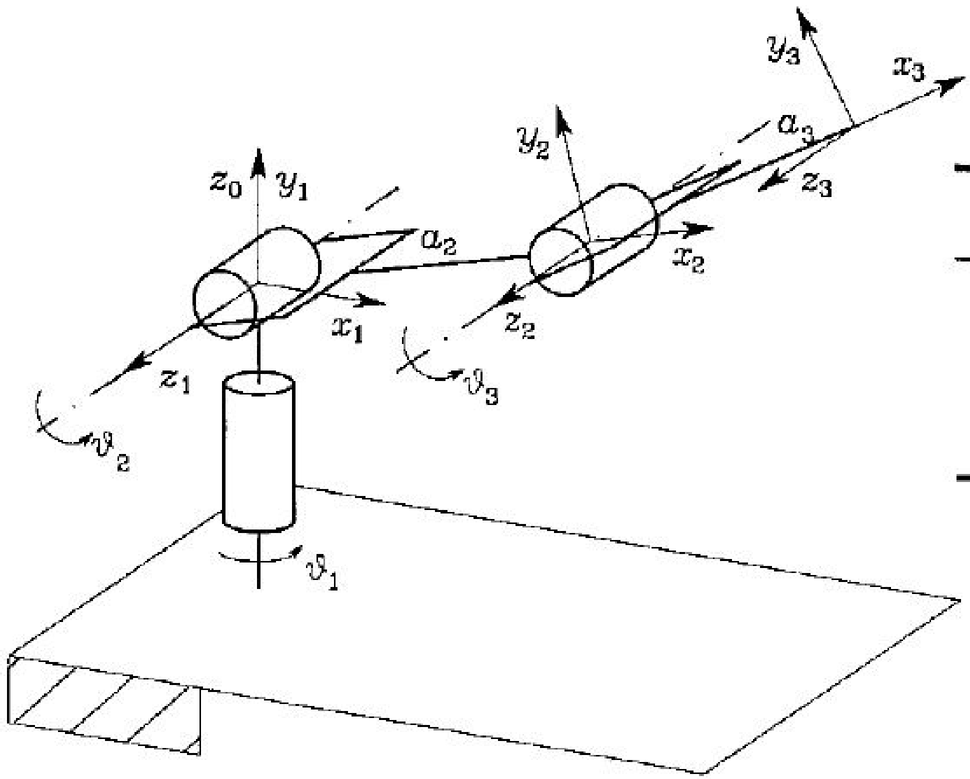


$$A_1^0(\vartheta_1) = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad A_2^1(\vartheta_2) = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3^2(d_3) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

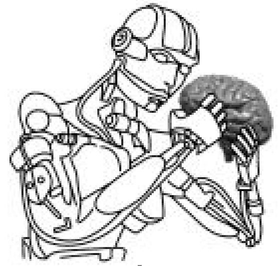
$$T_3^0(q) = A_1^0 A_2^1 A_3^2 = \begin{bmatrix} c_1 c_2 & -s_1 & c_1 s_2 & c_1 s_2 d_3 - s_1 d_2 \\ s_1 c_2 & c_1 & s_1 s_2 & s_1 s_2 d_3 + c_1 d_2 \\ -s_2 & 0 & c_2 & c_2 d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Examples: Anthropomorphic Arm



Link	a_i	α_i	d_i	ϑ_i
1	0	$\pi/2$	0	ϑ_1
2	a_2	0	0	ϑ_2
3	a_3	0	0	ϑ_3

Examples:
Anthropomorphic Arm

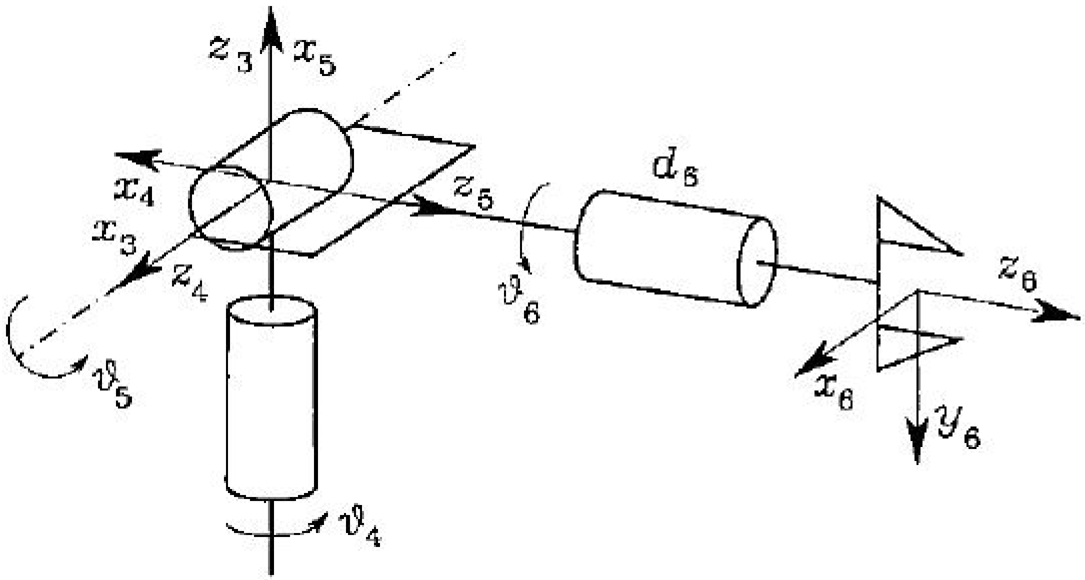


$$A_1^0(\vartheta_1) = \begin{bmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_i^{i-1}(\vartheta_i) = \begin{bmatrix} c_i & -s_i & 0 & a_i c_i \\ s_i & c_i & 0 & a_i s_i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad i = 2, 3$$

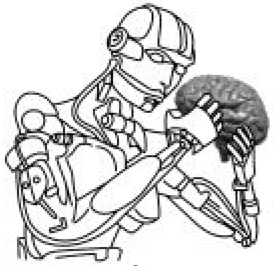
$$T_3^0(q) = A_1^0 A_2^1 A_3^2 = \begin{bmatrix} c_1 c_{23} & -c_1 s_{23} & s_1 & c_1(a_2 c_2 + a_3 c_{23}) \\ s_1 c_{23} & -s_1 s_{23} & -c_1 & s_1(a_2 c_2 + a_3 c_{23}) \\ s_{23} & c_{23} & 0 & a_2 s_2 + a_3 s_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Examples:
Spherical Wrist



Link	a_i	α_i	d_i	ϑ_i
4	0	$-\pi/2$	0	ϑ_4
5	0	$\pi/2$	0	ϑ_5
6	0	0	d_6	ϑ_6

Examples:
Spherical Wrist

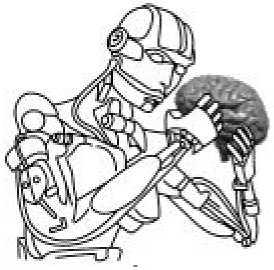


$$A_4^3(\vartheta_4) = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad A_5^4(\vartheta_5) = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

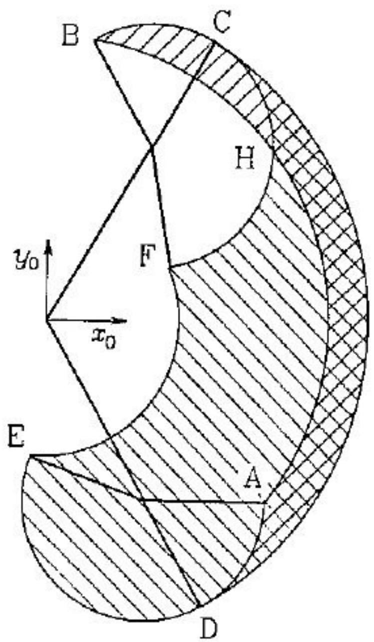
$$A_6^5(\vartheta_6) = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

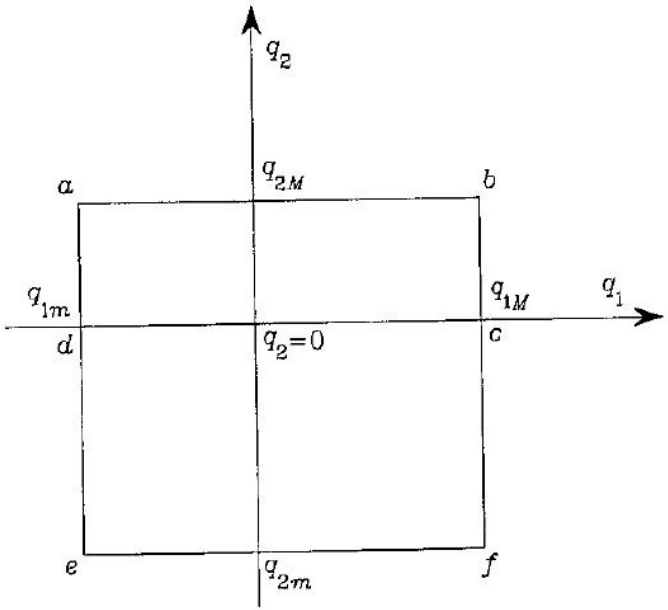
$$T_6^3(q) = A_4^3 A_5^4 A_6^5 = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 & c_4s_5d_6 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 & s_4s_5d_6 \\ -s_5c_6 & s_5s_6 & c_5 & c_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Workspace Considerations

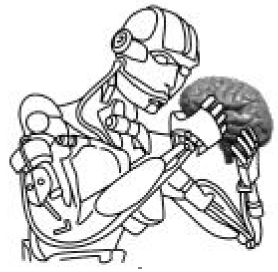


- *Operational space* – set of all configurations a end-effector can achieve
  - Orientation and position
- *Joint space* – possible values the joints can take





Workspace Considerations



- *Accuracy* – Deviation between assigned position and actual position of end-effector

- *Repeatability* – Ability to return to previous position

- *Kinematic Redundancy* – Number of degrees greater than the number of variables necessary to describe task
  - Multiple configurations to reach same end-effector position